Exam problems for the course
Introduction to General Relativity
fall semester 2006.

These are hand in assignments for the course in "Introduction to General Relativity" given at Masaryk University at the fall semester 2006. They consist the first part of the course requirements, the second part being an oral exam. The solutions to the problems should be handed in minimum one week before the oral exam. The answers to the problems can be written in English or Czech, they can be written by hand or on the computer but they should be legible. **Do not leave out any part of the calculation! Motivate your assumptions and approximations carefully**. As a minimum requirement to pass the course I have 38 points but more points of course gives higher grades. Please observe that if you hand in just enough problems to get 38 points, chances are that you will have some mistake somewhere and then you will not pass the exam!

1. Consider the metric

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 + dz^2 \tag{1}$$

A particle moves in this space along a parametrized curve given by

$$\begin{aligned} t &= \lambda \\ \rho &= R \\ \phi &= \omega\lambda \\ z &= 0 \end{aligned} \tag{2}$$

What is the four velocity of the particle? Are there any physical constraints on the constants ω and R? What is the four acceleration that the particle feels? (**8p**)

2. Confirm by explicit calculation that the Christoffel symbols constructed as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\gamma} \left(\partial_\mu g_{\nu\gamma} + \partial_\nu g_{\mu\gamma} - \partial_\gamma g_{\mu\nu} \right), \tag{3}$$

transform correctly (as a connection, not as a tensor) under a coordinate transformation $x^\mu \to x'^\mu(x)$. (**10p**)

3. For a tree dimensional metric of the form

$$ds^2 = h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3^2 dx_3^2, \tag{4}$$

where h_i are general functions of all the x coordinates, find explicit expressions for the gradient and Laplacian acting on scalars and also for the divergence operator acting on vectors. Compare your result to the known formulas for spherical and cylindrical coordinates. You may use that the expressions for these operators in term of the covariant derivative are $\nabla_\mu f$, $\nabla^\mu \nabla_\mu f$ and $\nabla_\mu V^\mu$ but remember that the "usual" formulas are expressed in terms of an orthonormal basis and not a coordinate basis! (**10p**)

4. For the two dimensional space $M = \{(x,y) : y > 0\}$ equipped with the metric

$$ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right), \tag{5}$$

compute all geodesics and their lengths. Compute also all components of the Riemann tensor as well as the Riccitensor and the Ricciscalar. (**10p**)

5. Elements in the chromosphere of the Sun emit sharp spectral lines. A student in relativity observes one such known line in a spectrometer here on Earth. According to general relativity, the emitted light is affected by the mass of the Sun. (You may ignore the influence of the gravitational field of the earth in this problem.) Calculate, using the general theory of relativity and to lowest order in the gravitational constant, the magnitude and sign of the relative frequency shift $\frac{\Delta\nu}{\nu}$ of this spectral line. the solar mass is about $2.0 \cdot 10^{30}$ kg. Newton's gravitational constant is $G \approx 6.7 \cdot 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}$. The solar radius is about $7.8 \cdot 10^8$ m and the average Sun-Earth distance is about $1.5 \cdot 10^{11}$ m. (**7p**)

6. A Killing vector is a vector field satisfying the equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{6}$$

Show that any Killing vector satisfies the equation

$$\nabla^\nu \nabla_\nu \xi^\mu + R^\mu{}_\nu \xi^\nu = 0 \tag{7}$$

(**5p**)

7. A good approximation to the metric outside the surface of the Earth is provided by

$$ds^2 = -\left(1 - \frac{2MG}{r}\right) dt^2 + \left(1 + \frac{2MG}{r}\right) dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{8}$$

where the gravitational potential $-\frac{GM}{r}$ may be assumed to be small. Here G is Newton's constant and M is the mass of the Earth.

   (a) Imagine that there is a clock on the surface of the Earth at distance R_1 and another clock on a tall building at distance R_2 from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time t. Which clock moves faster?

   (b) Solve for a geodesic corresponding to a circular orbit around the equator of the Earth ($\theta = \frac{\pi}{2}$). What is $\frac{d\phi}{dt}$?

   (c) How much proper time elapses while a satellite at radius R_1 (skimming along the surface of the earth,neglecting air resistance etc.) completes one orbit? (You can work to first order in the gravitational potential if you like.) Plug in the actual numbers for the radius of the Earth and so on to get an answer in seconds. How does this number compare to the proper time elapsed on the clock stationary on the surface?

(**10p**)

8. Consider a radial null geodesic propagating in a Robertson-Walker cosmology. Show that in the dust filled closed universe, a light ray emitted at the big bang travels precisely all the way around the universe by the time of the "big crunch". Show also that in the radiation filled closed universe, a light ray emitted at the big bang travels precisely halfway around the universe the time of the big crunch. Hint: it is useful to use a coordinate ψ related to the radius r as

$$\psi = \arcsin r \tag{9}$$

(**15p**)