Exam problems for the course
Statistical physics and thermodynamics
fall semester 2006.

These are hand in assignments for the course in "Statistical physics and thermodynamics" given at Masaryk University at the fall semester 2006. They consist the first part of the course requirements, the second part being an oral exam. The solutions to the problems should be handed in minimum one week before the oral exam. The answers to the problems can be written in English or Czech, they can be written by hand or on the computer but they should be legible. **Do not leave out any part of the calculation! Motivate your assumptions and approximations carefully**. As a minimum requirement to pass the course I have 24 points but more points of course gives higher grades. Please observe that if you hand in just enough problems to get 24 points, chances are that you will have some mistake somewhere and then you will not pass the exam!

1. Consider a system consisting of two particles, each of which can be in any one of three quantum states of respective energies, 0, ϵ and 3ϵ. The system is in contact with a heat reservoir at temperature T. (**2p**)
   (a) Write an expression for the partition function Z if the particles obey classical MB statistics and are considered distinguishable.
   (b) What is Z if the particles obey BE statistics?
   (c) What is Z if the particles obey FD statistics?
2. A simple harmonic one-dimensional oscillator has energy levels given by $E_n = (n + \frac{1}{2})\hbar\omega$, where ω is the characteristic frequency of the oscillator and the quantum number n can assume the possible integral values $n = 0, 1, 2, 3, \ldots$. Suppose that such an oscillator is in thermal contact with a heat reservoir at temperature T. (**3p**)
   (a) Find the ratio of the probability of the oscillator being in the first excited state to the probability of its being in the ground state.
   (b) Find the mean energy of the oscillator as a function of the temperature T.
3. For ideal gases in two dimensions, find (**6p**)
   (a) The heat capacity at constant area in the high-temperature limit for both the Fermi and Bose cases.
   (b) The heat capacity at constant area in the low-temperature limit for the Fermi case.
4. Find the high- and low-temperature limits of the heat capacity of a Debye solid in two dimensions. (**4p**)
5. A system consists of N very weakly interacting particles at a temperature T sufficiently high so that classical statistical mechanics is applicable. Each particle has mass m and is free to perform one-dimensional oscillations about its equilibrium position. Calculate the heat capacity of this system or particles at this temperature in each of the following cases: (**4p**)

(a) The force effective in restoring each particle to its equilibrium position is proportional to its displacement x from this position.

(b) The restoring force is proportional to x^2.

6. Assume the following highly simplified model for calculating the specific heat of graphite, which has a highly anisotropic crystalline layer structure. Each carbon atom in this structure can be regarded as performing simple harmonic oscillations in three dimensions. The restoring forces in directions parallel to a layer are very large; hence the natural frequencies of oscillations in the x and y directions lying within the plane of a layer are both equal to a value $\omega_{||}$ which is so large that $\hbar\omega_{||}$ is much greater than the temperature corresponding to $300K$. On the other hand, the restoring force perpendicular to a layer is quite small; hence the frequency of oscillation $\omega_{\perp}$ of an atom in the z direction perpendicular to a layer is so small that $\hbar\omega_{\perp}$ is much smaller than the energy corresponding to $300K$. On the basis of this model, what is the molar specific heat (at constant volume) of graphite at $300K$. (**4p**)

7. Electromagnetic radiation at temperature T_i fills a cavity of volume V. If the volume of the thermally insulated cavity is expanded quasi statically to a volume $8V$, what is the final temperature T_f? (**4p**)

8. Use the Debye approximation to find the equation of state for a solid; i.e. find the pressure $\bar{p}$ as a function of V and T. What are the limiting cases valid when $T \ll \theta_D$ and when $T \gg \theta_D$? Express your answer in terms of the quantity

$$\gamma \equiv -\frac{V}{\theta_D}\frac{d\theta_D}{dV} \tag{1}$$

Assume that γ is a constant, independent of temperature. (It is called the Grüneisen constant.) Show that the coefficient of thermal expansion α is then related to γ by the relation

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = \kappa\left(\frac{\partial p}{\partial T}\right)_V = \kappa\gamma\frac{C_V}{V} \tag{2}$$

where C_V is the heat capacity of the solid and κ is the compressibility. (**4p**)

9. Assume the existence of a Bose gas with dispersion relation $E = A|k|^n$ where n is any natural number. If the number of particles is not conserved, compute the dependence of the specific heat C_V on the temperature T. (**4p**)

10. Assume that we have a classical ideal gas where the particles also carry an internal degree of freedom. So apart from carrying kinetic energy $p^2/2m$ they also carry internal energy $\pm\Delta$. Show how one could measure delta by measuring the heat capacity. (**4p**)

11. Calculate the magnetic susceptibility of a free electron gas! In an electron gas there two competing effects that will decide how the induced magnetic

field will be when one applies an external magnetic field. The electrons themselves carry spin to which there is a magnetic moment associated. The magnetic moments tend to align with the magnetic field thus creating an induce magnetic moment in the same direction as the applied magnetic field. This is *paramagnetic* behavior. However, since the electrons are themselves charged they will move in circles in the magnetic field which will create a current that tends to reduce the applied external magnetic field. This is *diamagnetic* behavior. Calculate the Landau potential for these two problems independently and calculate the susceptibility χ it gives rise to according to the formula

$$M = -\left(\frac{\partial \Omega}{\partial H}\right)_{T,V,\mu} \tag{3}$$

$$\chi = \frac{\partial M}{\partial H} = -\frac{\partial^2 \Omega}{\partial H^2} \tag{4}$$

Start from the formula for the Landau potential using the ideal gas approximation

$$\Omega = -T \sum_a \ln\left(1 + e^{(E_a - \mu)/T}\right) \tag{5}$$

In the paramagnetic case, the states have different energy according to if the spin is up or down

$$E_a = \frac{\mathbf{p}^2}{2m} \pm \beta H \tag{6}$$

where $\beta = \frac{|e|\hbar}{2mc}$ is the Bohr magneton and H is the external magnetic field. In the diamagnetic case, as was shown in class, the sum over states can be exchanged with

$$\sum_a \to \sum_{n=0}^{\infty} \int dp_z 2 \frac{V}{(2\pi\hbar)^2} \frac{|e|H}{c} \tag{7}$$

where the energy of the states is given by

$$E = \hbar\omega(n + \frac{1}{2}) + \frac{p_z^2}{2m} \tag{8}$$

To get explicit results, use the high temperature approximation to lowest order and use what you know for the free electron gas without a magnetic field. Is the gas paramagnetic or diamagnetic? (**8p**)