Hand-in assignments in Quantum Mechanics, spring semester 2004.

These are hand in assignments for the course in Quantum mechanics at the Masaryk University in the spring of year 2004. They are the first part of the requirement of the course, the second being an oral exam. The problems should be handed in minimum one week before the oral exam. **Do not leave out any part of the calculations and motivate your assumptions and approximations carefully**. You my answer in Czech *or* English. The required minimum number of points is **30**.

Propagators and Path Integrals

1. A model of a moving wave-packet in 1 dimension is given by the wave-function

   $$N \int dp e^{-\frac{a}{2}(p-p_0)^2} |p\rangle$$

   where a is a constant and N is the normalization factor. Determine N and use the propagator of a free particle to find how the packet moves in time. *Interpret* your result! **(3p)**

2. Find the momentum space propagator $\langle p', t'|p, t\rangle$ for a free 1 dimensional particle using the time evolution operator. Show how to get the configuration space propagator

   $$\langle x', t'|x, t\rangle = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} e^{\frac{im}{2\hbar}\frac{(x'-x)^2}{\Delta t}}$$

   from the expression from the momentum space propagator. **(3p)**

3. Use the identity

   $$\left(\frac{1}{\sqrt{1-\zeta^2}}\right) \exp\left[\frac{-(\xi^2+\eta^2-2\xi\eta\zeta)}{(1-\zeta^2)}\right]$$
   $$= \exp\left[-(\xi^2+\eta^2)\right] \sum_{n=0} \left(\frac{\zeta^n}{2^n n!}\right) H_n(\xi) H_n(\eta)$$

   to find the propagator for the 1 dimensional harmonic oscillator using the "old" method (i.e. express the initial wave-function in terms of energy eigenfunctions and then use that the time evolution for these states is trivial). **(4p)**

4. Find the time evolution of the 1 dimensional harmonic oscillator state

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega(x-x_0)^2}{2\hbar}},$$

using the harmonic oscillator propagator

$$\begin{aligned} K(x', t'; x, t) &= \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega\Delta t}} \times \\ &\exp\left\{\left(\frac{im\omega}{2\hbar\sin\omega\Delta t}\right)\left((x'^2 + x^2)\cos\omega\Delta t - 2x'x\right)\right\} \end{aligned}$$

Interpret your result! **(5p)**

5. The propagator for a charged particle in a homogeneous electric field E can be written as

$$K(x', t'; x, t) = \sqrt{\frac{m}{2\pi i\hbar\,(t'-t)}} \times$$
$$\exp\left\{\frac{im}{2\hbar}\frac{(x'-x)^2}{t'-t} + \frac{iE}{2\hbar}(x'+x)(t'-t) - \frac{i}{24}\frac{E^2}{\hbar m}(t'-t)^3\right\}.$$

Find the time evolution of a wavepacket

$$\psi(x, t=0) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} e^{-x^2}.$$

Interpret your result! Does it agree with your intuition? **(5p)**

Angular momentum

1. The Hamiltonian for a spin 1 system is given by

$$\hat{H} = A\hat{J}_z^2 + B\left(\hat{J}_x^2 - \hat{J}_y^2\right).$$

Solve this problem *exactly* to find the normalized energy eigenstates and eigenvalues. A spin dependent Hamiltonian of this kind actually appears in crystal physics. **(2p)**

2. An angular momentum eigenstate $|j, m = m_{\max} = j\rangle$ is rotated by an infinitesimal angle ϵ about the y-axis. By using the angular momentum operators, obtain an expression for the probability for the new rotated state to be found in the original state up to terms of order ϵ^2. **(2p)**

3. An atom has total angular momentum $j = \frac{3}{2}$ and z-component $m = \frac{3}{2}$. Calculate the probabilities to find the system with any particular angular momentum component along an axis leaning at an angle θ with respect to the z-axis. **(4p)**

4. A particle in a spherically symmetrical potential is known to be in an eigenstate of $\mathbf{L}^2$ and L_z with eigenvalues $\hbar^2 l(l+1)$ and $m\hbar$, respectively. Prove that the expectation values between $|lm\rangle$ states satisfy

$$\langle L_x \rangle = \langle L_y \rangle = 0, \quad \langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{l(l+1)h^2 - m^2\hbar^2}{2}$$

   (2p)

5. Calculate the explicit form of the spin 1 representation of the rotation operator $\hat{R}_x(\phi)$ in two ways. First by exponentiating the explicit form of $\hat{J}_x$ and second by combining the rotation matrices of two spin $\frac{1}{2}$ particles. **(4p)**

6. a) Evaluate

$$\sum_{m=-j}^{j} m \left| \langle j, m | \, \hat{R}_y(\theta) \, | j, m' \rangle \right|^2 ,$$

   for *any* j; then check your answer for $j = \frac{1}{2}$.

   b) Prove, for any j, that

$$\sum_{m=-j}^{j} m^2 \left| \langle j, m | \, \hat{R}_y(\theta) \, | j, m' \rangle \right|^2 = \frac{1}{2} j(j+1) \sin^2\theta + \frac{m'^2}{2} \left(3\cos^2\theta - 1 \right).$$

   (4p)

7. We are to add angular momenta $j_1 = 2$ and $j_2 = \frac{1}{2}$ to form $j = \frac{3}{2}$ and $\frac{5}{2}$ states. Express all (ten) j, m eigenkets in terms of $|j_1 j_2; m_1 m_2\rangle\rangle$. **(4p)**

8. The wave function of a particle subjected to a spherically symmetrical potential $V(r)$ is given by

$$\psi(\mathbf{x}) = (x + y + 3z) f(r).$$

a) Is ψ an eigenfunction of $\mathbf{L}^2$? If so, what is the l-value? If not, what are the possible values of l we may obtain when $\mathbf{L}^2$ is measured?

b) What are the probabilities for the particle to be found in various m_l states?

c) Suppose it is known somehow that $\psi(\mathbf{x})$ is an energy eigenfunction with eigenvalue E. Indicate how we may find $V(r)$.

(4p)

9. Consider a spin-less particle bound to a fixed center by a central force potential. Relate, as much as possible, the matrix elements

$$\langle n', l', m'| \mp \frac{1}{\sqrt{2}}(x \pm iy) |n, l, m\rangle \text{ and } \langle n', l', m'| z |n, l, m\rangle$$

using only the Wigner-Eckart theorem. Make sure to state under what conditions the matrix elements are non-vanishing. **(6p)**

10. The expectation value

$$Q = e \langle j, m = j| (3z^2 - r^2) |j, m = j\rangle ,$$

is known as the quadrupole moment. Evaluate

$$\langle j, m'| (x^2 - y^2) |j, m = j\rangle ,$$

(where $m' = j, j-1, \ldots$) in terms of Q and appropriate Clebsch-Gordan coefficients. **(6p)**

Scattering theory

1. Determine, using the Born approximation, the differential and the total scattering cross-section in the low energy limit for a spherical potential well

$$V = \begin{cases} -|V_0| & \text{for} \quad r < a \\ 0 & \text{for} \quad r > a. \end{cases}$$

(4p)

2. Determine in the Born approximation the differential and the total scattering cross-section for the potential $V = \frac{V_0}{r} e^{-\frac{r}{a}}$. **(4p)**

3. Consider a potential

$$V = \begin{cases} 0 & \text{for} \quad r > R \\ V_0 & \text{for} \quad r < R \end{cases}$$

where V_0 is a positive *or* negative constant. Using the method of partial waves, show that for $|V_0| \ll E = \frac{\hbar^2 k^2}{2m}$ and $kR \ll 1$ the differential cross section is isotropic and that the total cross section is given by

$$\sigma_{\text{tot}} = \left(\frac{16\pi}{9}\right) \frac{m^2 V_0^2 R^6}{\hbar^4}.$$

Suppose the energy is raised slightly. Show that the angular distribution can then be written as

$$\frac{d\sigma}{d\Omega} = A + B\cos(\theta)$$

Obtain an approximate expression for $\frac{B}{A}$. **(6p)**

4. Use the method of partial waves to show that in scattering of low-energy particles on a spherical potential well of depth $-V_0$ there is a special value of V_0 for which the phase shift of the $l = 0$ partial wave is π while higher-order phase shifts are negligibly small. What happens to the scattering cross-section in this case? This effect was observed by Ramsauer in the scattering of low-energy (0.7 eV) electrons by rare-gas atoms. Using an atomic radius of 10^{-10} m, what must be the depth ($= V_0$) of the effective potential well for helium, to explain the observations of Ramsauer? **(6p)**

5. Consider the scattering of a particle by a repulsive delta function shell potential

$$V(r) = \frac{\hbar^2\gamma}{2m}\delta(r - R),$$

   a) Set up an equation that determines the s-wave phase shift δ_0 as a function of k (remember that $E = \frac{\hbar^2 k^2}{2m}$).
   b) Assume now that γ is very large,

$$\gamma \gg \frac{1}{R}, k.$$

Show that if $\tan kR$ is *not* close to zero, the s-wave phase shift resembles the hard-sphere result discussed in the lectures. Show also that for $\tan kR$ close to (but not exactly equal to) zero, resonance behavior is possible; that is, $\cot \delta_0$ goes through zero from the positive side as k increases. Determine approximately the positions of the resonances keeping terms of order $\frac{1}{\gamma}$.

(6p)

Relativistic Quantum Mechanics

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