

# More-particle Correlations in Fermion Beams

Dissertation

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# Vícečásticové korelace ve fermionových svazcích

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# Chapter 1

## Introduction

The aim of this dissertation is to describe some interesting aspects of more-particle correlations in fermion beams that occur due to the indistinguishableness of fermions. A considerable part of this dissertation is inspired by the methods and results of optical theory of coherence that deals with analogous phenomena in photon beams. For this reason it is convenient to quote at the beginning the book *Optical Coherence and Quantum Optics* by L. Mandel and E. Wolf [1] (Chapter 4, p. 147), where a brief introduction to the optical theory of coherence is given:

It seems hardly necessary to stress that every electromagnetic field found in the nature has some fluctuations associated with it. Even though these fluctuations are, as a rule, much too rapid to be observed directly, one can deduce their existence from suitable experiments that provide information about correlations between the fluctuations at two or more space-time points.

The simplest manifestation of correlations in optical fields are the well-known interference effects that arise when two light beams that originate from the same source are superposed. With the ability of modern light detectors and electronic circuitry of very short resolving time, other types of correlations in optical fields began to be studied in more recent times. These investigations, as well as the development of lasers and other novel types of light sources, led to a systematic classification of optical correlation phenomena and the complete statistical description of optical fields. The area of optics concerned with such questions is now generally known as optical coherence theory.

The optical coherence theory is nowadays a very important part of physics and covers a vast area of all kinds optical phenomena. In comparison to this, the theory of coherence of free fermion fields suffers a considerable delay. This fact is not difficult to understand and it has two main reasons.

First, the experimental support in both the cases can be hardly compared, which is clear already from the fact that we meet optical phenomena in the everyday life, which cannot be said about fermion beams (an exception is a television screen; however, it hardly enlarges our direct experience with fermions). For example, we can look at a bulb filament

through a piece of fabric to see wonderful diffraction effects or observe colours on a water puddle with an oil drop in it. It is also quite easy to get a coherent light beam — just by letting the light from a star pass through an interference filter of a narrow bandwidth we obtain light that is spatially and temporally very coherent. On the other hand, to obtain a coherent neutron beam one needs a nuclear reactor, and to get a coherent electron beam an ultra-high vacuum chamber with an extremely sharp field-emission tip is necessary. Moreover, the coherence of so obtained electron or neutron beams can never reach the quality of the above mentioned filtered light from a star.

Second, there is one fundamental difference between photon and fermion fields: in contrast with light, there is no classical wave theory for fermions. Many optical coherence and correlation phenomena such as interference fringes or the Hanbury-Brown and Twiss effect can be understood without quantum mechanics, just with the help of the classical electromagnetic theory. This enabled the development of the optical coherence theory already long before quantum mechanics appeared and on this base the quantum coherence theory could then develop rapidly. In contrast to this, there is no classical wave theory for fermions and it is completely unavoidable to use the quantum theory for describing coherence and correlation phenomena that occur in fermion beams.

I would like to contribute to reducing the above mentioned contrast by this dissertation. Because of concreteness, I will concentrate on the electrons, but most of the results to be obtained hold also for other types of fermions. The main reason for choosing the electrons is the experimental background at the Institute of Applied Physics at the University of Tübingen, Germany, where currently an electron coincidence experiment is going on. Therefore I will also consider free electrons in this dissertation that seem to have attracted up to now less theoretical and experimental interest than the electrons in condensed matter.

The dissertation is organized as follows: first, in Chapter 2 the basic elements of an electron correlation experiment are described. In Chapter 3 the electron correlation functions are defined with the help of which the coherence and correlational properties of the electron fields will be described. In Chapter 4 some aspects of electron field operators are discussed. In Chapter 5 the chaotic state of the electron field is introduced which is supposed to describe the free electrons in the best way, and also some other alternatives of electron states are given. Then rather a complicated Chapter 6 follows where the correlation functions are evaluated for the states defined in Chapter 5 without considering the electron spin. This imperfection is improved in the next Chapter 7, where also the spin is taken into account. Thanks to a determinant form of the correlation function of the chaotic state, it is possible in Chapter 8 to apply the methods of linear algebra to show an important inequality between different correlation functions. In the next Chapter 9 the properties of two-electron correlation function are discussed in relation to experiment. The influence of an electron biprism and a Wien filter on the correlation function is discussed in Chapter 10. In Chapter 11 an interesting aspect of the correlation functions is discussed that can seem to lead to senseless results. In the last Chapter 12 the dissertation is summarized and the possible directions of further research are outlined.

I must warn the reader that some parts of the dissertation are quite complicated and therefore maybe less readable. This holds especially about Chapter 6 that can be skipped

except for the fundamental results expressed in Eqs. (6.28) and (6.30). Chapter 8 is written in a formal mathematical way and is rather complicated. However, I do not recommend the reader to skip it completely because I think that it is even an esthetic experience to see how the linear algebra beautifully works when applied on the correlation functions.

# Chapter 2

## Experiment

Even though this dissertation is a theoretical one, I think it is important not to lose connection with the experiment because it is basically the experiment that enables to verify if a physical theory is correct or not. Therefore I include this chapter at the beginning of this dissertation. Moreover, it will be useful to mention different components used in the experiments with free electrons at this place because I will often speak about them later. So, I will describe now briefly a general free-electron experimental setup and its most important parts and then explain the idea of the electron coincidence experiment that is currently running in Tübingen.

### 2.1 Basic components of an experiment with free electrons

The basic components of any free-electron experiment are the electron source and one or more detectors. Then there can be also additional elements that modify the electron field according to the aim of the experiment. Such elements are for example an electrostatic biprism, a Wien filter, an electrostatic quadrupole, a coil etc. I will mention just the ones that are of particular importance. The scheme of an electron interferometer including a photograph of a real interferometer used in Tübingen is shown in Fig. 2.1, a more detailed description can be found in [2].

#### Electron source

There are several methods of obtaining free electrons that are mostly based on emission of electrons from metal. The most important two of them seem to be the thermal emission and the field emission.

In the case of thermal emission, the electrons get the energy to leave their bound state in metal from thermal excitations. The energy bandwidth of the emitted electrons is a few electronvolts. A thermal source does not provide a very coherent electron field and is thus not suitable for electron correlation experiments.



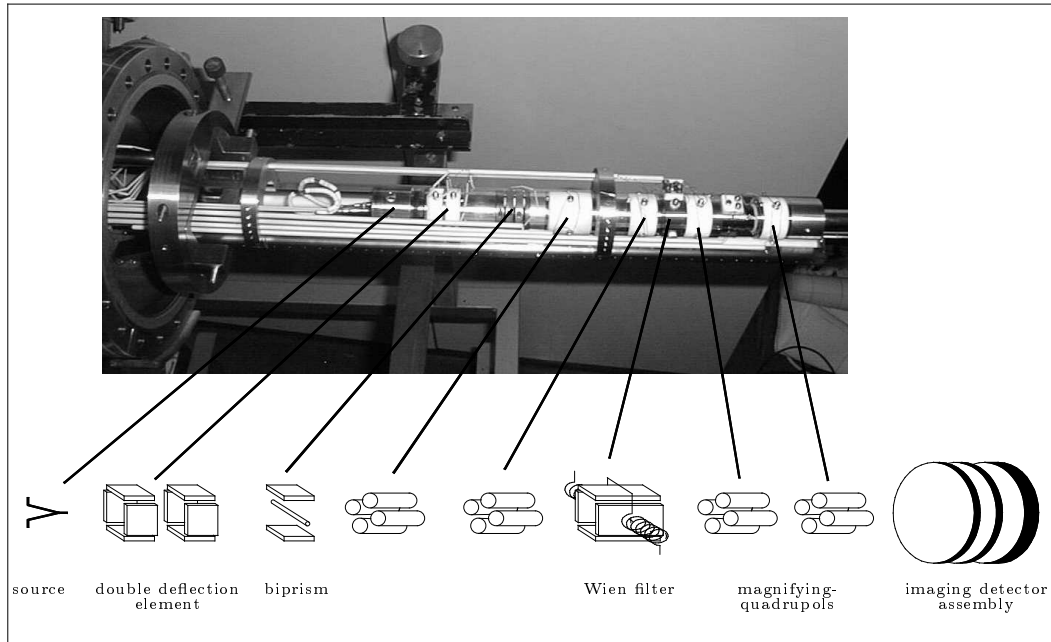


Figure 2.1: The electron interferometer used in Tübingen

A field-emission source is a very sharp tungsten tip with a diameter of about 50 nm at the top and with a negative potential of 1 to 1000 kV. A very strong electrostatic field that is built up near the tip “picks” the electrons from the metal. More precisely said, the region where the potential energy of the electrons is less than in the metal is so near to the tip that the electrons can tunnel from the metal to vacuum [3]. The energy bandwidth in this case is a fraction of electronvolt, which makes the field-emission sources the most coherent electron sources known up to now and therefore very suitable for interferometric experiments. I will concentrate on them in this dissertation.

I should also mention an europium-sulfide source that is able to emit partially polarized electrons when cooled to very low temperatures [4]. This source is a subject of extensive study.

## Electron detectors

The temporal technology enables to detect single electrons. This is achieved by using an image intensifier that produces about a million secondary electrons if an electron hits it. The image intensifier consists of usually two cascaded microchannel plates placed immediately after one another. The microchannel plate is a honeycomb-like regular structure of several millions of tiny tubules with a diameter of six micrometers and a length of half a millimeter. Each such tube works as a secondary-electron multiplier thanks to a semi-conducting material that it is from inside covered with. Thus if an electron hits the plate,

about a thousand electrons leave the plate on the opposite side. If two plates after one another are used, about a million electrons are obtained that can be detected either by a fluorescent screen and observed optically, or collected by a detection anode. In the latter case a small negative electric pulse is obtained that can be further processed.

## Electrostatic biprism

The basic principle of all interferometers is that the particles (e.g. photons or electrons) can come from the source to the detector by at least two different paths and that interference occurs between the partial beams coming by these paths. In optics, there are many methods to achieve this that can be divided into two groups: the methods using the so-called amplitude splitting and the ones using the wave front splitting. The former principle is used e.g. in Michelson, Fabry-Perot or Jamin interferometers, the latter one in Fresnel biprism or Fresnel mirrors interferometer and in the Young experiment [5]. The main element in the experiments that use the amplitude splitting is a beam splitter. Experiments with amplitude splitting of electrons on crystals have not been very successful and there is no practically usable beam splitter for free electrons, even if several types of such splitters are known for a two-dimensional electron gas [6, 7, 8, 9]. On the other hand, the situation with the wavefront splitting is much better. An element analogous to the Fresnel biprism in optics has been discovered for electrons in 1954 by G. Möllenstedt, namely, an electrostatic biprism consisting of a positively charged conducting filament that attracts electrons by the Coulomb force [10]. This has made interference experiments with electrons possible such as observing interference fringes [10, 11], measurements of the longitudinal coherence [12], measurement of the energy spectrum by Fourier spectroscopy [13] and also very practical applications such as electron holography [14].

## Wien filter

A Wien filter consists of homogeneous electrostatic and magnetostatic fields whose intensities are perpendicular to each other and to the optical axis of the experiment [12]. The intensities of the electric and magnetic fields are chosen in such a way that the electric and magnetic forces acting on the electrons cancel each other. Now, if there are two electron beams passing through the filter at places with different electrostatic potentials, the beam passing at the higher potential is accelerated with respect to the beam passing the filter at the lower potential. Therefore the Wien filter has the same effect as if a longitudinal shift between the beams were introduced. However, the mutual phase of the beams remains unchanged because the phase velocity of the electrons, in contrast to the group velocity, does not change in the Wien filter. This is a partial analogy to the situation in optics when a medium with a different index of refraction inserted to the way of one beam. However, in the latter case there is of course also a phase difference, which is not the case of electrons in the Wien filter.

## Electrostatic quadrupole

An electrostatic quadrupole is used for spreading the very narrow beam that is emitted by the field-emission source. Without this it would be very difficult to observe interference fringes with the microchannel plate image intensifier and impossible to illuminate detectors with the size of a few millimeters coherently.

## 2.2 Electron coincidence experiment

The basic setup of an electron coincidence experiment that is designed to measure two-electron correlations is simple: there is basically an electron source, two detectors that are able to detect single electrons, and an electronic apparatus for processing the signals from the two detectors. The measurement principle can be imagined in the following naive way: If an electron comes to the first detector (or “start detector”), a very fast “stop-watch” is started. This watch then runs and is stopped as soon as another electron comes to the second detector (or “stop detector”). The time interval  $\tau$  that the stop-watch has measured is then stored, the watch is reset and the whole procedure is repeated. In this way, the statistic of the times intervals between electron arrivals to the first and second detectors is obtained.

In practice, the above described idea is realized as follows (see Fig. 2.2): The secondary

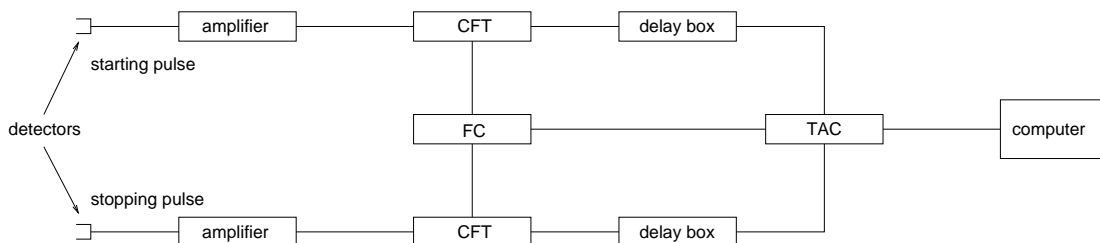


Figure 2.2: Scheme of the coincidence measurement setup

electrons from the microchannel plate are collected by two specially designed detectors and the resulting weak negative pulses travel by two coaxial cables to very fast microwave amplifiers. The amplified signals are then fed to two constant fraction triggers (CFT) that change each pulse into a well-defined right-angle pulse with a minimal raising time. The fast coincidence circuit (FC) selects only the pulse pairs that are separated by less than 10 ns because the ones with a larger separation are of no use for the experiment. The pulses can be then delayed to compensate the differences in the lengths of the coaxial cables and are sent to the hearth of the experiment, the time-to-amplitude converter (TAC). The TAC produces an output signal whose amplitude is proportional to the time difference  $\tau$  between the starting and stopping pulses. A multi-channel analyzer built in a computer, which is a kind of an A-D converter, then measures the amplitude of the signal from the TAC and gives it over to a computer program. After many such measurements are performed,

the statistics of the time differences  $\tau$  is obtained. The time resolution of the coincidence electronics alone is about  $10^{-11}$  s and of the entire experimental setup  $10^{-10}$  s.

The probability distribution of the time differences  $\tau$  between the starting and stopping pulses that is measured in this experiment is closely related to the joint probability of detecting an electron at the first detector and detecting another electron at the second detector  $\tau$  seconds later. We will call this probability the two-electron correlation function and, as we will see in the next chapter, it is a member of a large family of different correlation functions. The following chapters are devoted to evaluating such functions for field-emission electrons and investigating their properties.

# Chapter 3

## Correlation functions

Every electron field can be completely described by its density operator. However, the information contained in this operator is mostly too complex to be investigated directly. Usually we are interested only in some particular properties of the electron field and therefore introduce various physical quantities that describe these properties. Some of these quantities describe the coherence properties of the electron field and namely these are the ones in that we will be interested in this dissertation. As the physical laws by which the electrons are directed have a quantum-mechanical nature, all such quantities have only a statistical character. Of course it is reasonable to define these quantities in such a way that they can be measured experimentally, which enables a verification of the theory by the experiment. As all measurements with free-electron fields are based on a detection of an electron, it is clear that the quantities we are interested in will be closely connected with detection probabilities.

The most simple example of such a quantity is the probability density  $P(\mathbf{r})$  of finding an electron at some particular point  $\mathbf{r}$ . This probability density can express the coherence in some situations — for example a periodical varying of  $P(\mathbf{r})$  with the position  $\mathbf{r}$  (i.e. interference fringes) in an interferometer reflects the fact that the two partial beams are coherent because an interference between them occurs. However, the probability density  $P(\mathbf{r})$  cannot describe electron correlations because it deals with a one-electron process only (the detection of an electron). If we want to investigate correlational properties of an electron field, it is necessary to introduce quantities that involve more-electron detection processes. An example is the probability distribution of the number of electrons that arrive to a detector within a time interval  $T$  or the joint probability of registering electrons at different detectors and different moments of time. From the practical point of view, the latter quantity has a simpler structure than the former one and I will concentrate on it in this dissertation. The quantities based on the probabilities of multiple electron detections are called correlation functions.

### 3.1 Correlation functions as mean values of field operator products

Correlation function is a term familiar in all scientific areas where some statistical phenomena occur. Generally said, it is the mean value of a product of some fluctuating variables. In classical optics the correlation function expresses the statistical average of the product the complex analytic signals associated with some field variable (e.g. the intensity of the electric field) at different space-time points. In quantum optics, the operators corresponding to the complex analytic signal are the photon creation and annihilation operators. It is therefore natural to define the quantum-optical correlation function as the mean value of the product of the corresponding operators. As the photon operators do not mutually commute, however, different correlation functions can be defined using one set of the operators. There is a class of correlation functions with a particular importance, namely, the functions in which there is the same number of creation and annihilation operators and all the creation operators stand on the left-hand side from the annihilation operators (the so-called normal ordering). It can be shown that such correlation functions correspond to the probability of a multiple photon detection and that the normal ordering of the operators is closely connected with the fact that photons are absorbed at the detection process. The concept of electron correlation functions is completely analogous to the one of photons. As the electron detection processes are also based on an absorption of electrons, the correlation functions that come in question in this dissertation are the normally-ordered ones.

The normally-ordered operator of electron density at the point  $\mathbf{r}$  at the time  $t$  has the form

$$\hat{n}(\mathbf{r}, t) = \hat{\psi}^\dagger(\mathbf{r}, t)\hat{\psi}(\mathbf{r}, t), \quad (3.1)$$

where  $\hat{\psi}(\mathbf{r}_i, t_i)$  and  $\hat{\psi}^\dagger(\mathbf{r}_i, t_i)$  are the field operators annihilating and creating an electron at the space-time point  $(\mathbf{r}_i, t_i)$ , respectively<sup>1</sup>. Similarly, the normally-ordered operator of the probability density that an electron will be found at the point  $\mathbf{r}_1$  at time  $t_1$  and another electron will be found at the point  $\mathbf{r}_2$  at time  $t_2$  is

$$\hat{n}^{(2)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) = \hat{\psi}^\dagger(\mathbf{r}_1, t_1)\hat{\psi}^\dagger(\mathbf{r}_2, t_2)\hat{\psi}(\mathbf{r}_2, t_2)\hat{\psi}(\mathbf{r}_1, t_1). \quad (3.2)$$

Note the meaning of the normal ordering: if there is just one electron in the field, no two electrons can be found at the points  $(\mathbf{r}_1, t_1)$  and  $(\mathbf{r}_2, t_2)$ . Therefore the operator  $\hat{n}^{(2)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2)$  must give the zero mean value, which is really true thanks to its normal ordering. If the measurement were not destructive, this would not be necessary: we could register an electron at the point  $\mathbf{r}_1$  at the time  $t_1$  and then, at the time  $t_2$ , register the *same* electron at the point  $\mathbf{r}_2$ . The operator corresponding to such a non-destructive detection would be therefore no more normally ordered. This shows a narrow relation between the operator ordering and the detection mechanism.

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<sup>1</sup>if we say that some event (e.g. creation of an electron) happens at the space-time point  $(\mathbf{r}, t)$ , we mean that it happens at the point  $\mathbf{r}$  at the time  $t$

In a similar way, the operator corresponding to the probability density of finding  $k$  electrons at  $k$  points  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_k$  at times  $t_1, t_2, \dots, t_k$  can be written as follows:

$$\hat{n}^{(k)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2, \dots, \mathbf{r}_k, t_k) = \hat{\psi}^\dagger(\mathbf{r}_1, t_1) \cdots \hat{\psi}^\dagger(\mathbf{r}_k, t_k) \hat{\psi}(\mathbf{r}_k, t_k) \cdots \hat{\psi}(\mathbf{r}_1, t_1). \quad (3.3)$$

The  $k$ -electron correlation function<sup>2</sup> that expresses the probability density of finding  $k$  electrons at points  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_k$  and times  $t_1, t_2, \dots, t_k$  is defined as the quantum-mechanical average of the operator  $\hat{n}^{(k)}(\mathbf{r}_1, t_1, \dots, \mathbf{r}_k, t_k)$ :

$$\begin{aligned} G^{(k)}(\mathbf{r}_1, t_1, \dots, \mathbf{r}_k, t_k) &= \langle \hat{n}^{(k)}(\mathbf{r}_1, t_1, \dots, \mathbf{r}_k, t_k) \rangle \\ &= \langle \hat{\psi}^\dagger(\mathbf{r}_1, t_1) \cdots \hat{\psi}^\dagger(\mathbf{r}_k, t_k) \hat{\psi}(\mathbf{r}_k, t_k) \cdots \hat{\psi}(\mathbf{r}_1, t_1) \rangle. \end{aligned} \quad (3.4)$$

We will often use a shortened notation  $G_{12\dots k}^{(k)}$  instead of  $G^{(k)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2, \dots, \mathbf{r}_k, t_k)$ , with each number  $i$  in the subscript standing for one space-time point  $(\mathbf{r}_i, t_i)$ , and sometimes we will write simply  $G^{(k)}$  if it is clear to what points  $G^{(k)}$  corresponds. Similarly, we will denote the field operators  $\hat{\psi}(\mathbf{r}_i, t_i)$  and  $\hat{\psi}^\dagger(\mathbf{r}_i, t_i)$  simply as  $\hat{\psi}_i$  and  $\hat{\psi}_i^\dagger$ , respectively. If  $\hat{\rho}$  is the density operator of the electron field, in this shortened notation we can write

$$G_{12\dots k}^{(k)} = \langle \hat{\psi}_1^\dagger \cdots \hat{\psi}_k^\dagger \hat{\psi}_k \cdots \hat{\psi}_1 \rangle = \text{Tr} \{ \hat{\rho} \hat{\psi}_1^\dagger \cdots \hat{\psi}_k^\dagger \hat{\psi}_k \cdots \hat{\psi}_1 \} \quad (3.5)$$

## 3.2 Detection correlation function

An important remark has to be made at this place: the probability density  $\mathcal{G}_{12\dots k}^{(k)}$  of *detecting*  $k$  electrons at points  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_k$  and times  $t_1, t_2, \dots, t_k$  is not in general proportional to the  $k$ -electron correlation function defined above. In fact, the detection probability increases with the increasing velocity of electrons, even if the electron density remains constant<sup>3</sup>. This suggests that the detection probability is connected with particle fluxes rather than densities. However, it is not completely clear how this probability exactly looks like and how the “detection correlation function” should be defined. Some authors (e.g. Silverman and Saito [15, 16]) define it with the help of a kind of flux operators introduced by Goldberger *et al.* [17], which seems to be a good approximation. To find out the exact expression for the more-electron detection probability, it would be necessary to investigate

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<sup>2</sup>In literature there is not a unity in what is meant by the “correlation function of the  $k$ th order”. In quantum optics the order of the correlation function is defined by the number of field operators in the product that is averaged. In the electron theory, on the other hand, by a correlation function of the  $k$ th order is mostly understood the function containing  $2k$  field operators corresponding to the detection of  $k$  electrons (one operator pair for each electron). Therefore, to avoid misunderstanding, we will mostly speak about the  $k$ -electron correlation function in this work, which, in the language of quantum optics, has the order  $2k$ .

<sup>3</sup>This can be understood with the help of a simple analogy from the everyday life: if it rains, the probability that a drop hits our hand (a “detector”) during some particular time is proportional not only to the concentration of the drops in the air but also to the velocity with which they fall.

the interaction of the electron field with the detector on the microscopic level. This interaction is more complicated than e.g. the interaction of light with a detector in the case of a photodetection, one of the reasons being the indistinguishableness of the arriving electron with the electrons in the detector. The question of a microscopic theory of electron detection should deserve a proper analysis similar to the one given e.g. by Mandel and Wolf ([1], Chapter 14) for the photoelectric detection of light. We will not make this analysis here but only suggest the form of the more-electron detection probability heuristically.

It can be expected that the one-electron detection probability in a detector with an oriented cross-section  $\mathbf{S}$  and quantum efficiency  $\alpha$  is proportional to  $\alpha \langle \mathbf{S} \hat{\mathbf{j}} \rangle$ , where  $\hat{\mathbf{j}}$  is the flux operator at the point  $(\mathbf{r}, t)$ ,

$$\hat{\mathbf{j}} = \frac{i\hbar}{2m} \left\{ [\nabla \hat{\psi}^\dagger(\mathbf{r}, t)] \hat{\psi}(\mathbf{r}, t) - \hat{\psi}^\dagger(\mathbf{r}, t) \nabla \hat{\psi}(\mathbf{r}, t) \right\}. \quad (3.6)$$

The gradient of a field operator is defined as

$$\nabla \hat{\psi}(\mathbf{r}, t) = \nabla \hat{\psi}(x, y, z, t) = \left( \frac{\partial \hat{\psi}(x, y, z, t)}{\partial x}, \frac{\partial \hat{\psi}(x, y, z, t)}{\partial y}, \frac{\partial \hat{\psi}(x, y, z, t)}{\partial z} \right) \quad (3.7)$$

with

$$\frac{\partial \hat{\psi}(x, y, z, t)}{\partial x} = \lim_{h \rightarrow 0} \frac{\hat{\psi}(x + h, y, z, t) - \hat{\psi}(x, y, z, t)}{h} \quad \text{etc.} \quad (3.8)$$

The time probability density of registering an electron at each of the  $k$  detectors located at the points  $\mathbf{r}_1, \dots, \mathbf{r}_k$  at times  $t_1, \dots, t_k$  will be then

$$P_{12\dots k}^{(k)} = \left\langle : \prod_{i=1}^k \alpha_i \mathbf{S}_i \hat{\mathbf{j}}_i : \right\rangle, \quad (3.9)$$

the index  $i$  standing for each detector. Here  $: : \rangle$  denotes the normally-ordered product<sup>4</sup> and  $\nabla_i$  is the gradient with respect to  $\mathbf{r}_i$ . The detection correlation function then looks like

$$\mathcal{G}_{12\dots k}^{(k)} = \left\langle : \prod_{i=1}^k \hat{\mathbf{j}}_i : \right\rangle = \left\langle : \prod_{i=1}^k \frac{i\hbar}{2m} \left[ (\nabla_i \hat{\psi}_i^\dagger) \hat{\psi}_i - \hat{\psi}_i^\dagger \nabla_i \hat{\psi}_i \right] : \right\rangle. \quad (3.10)$$

and has a structure of a  $k$ -tensor (linear  $k$ -form) operating on the  $k$  vector variables  $\alpha_1 \mathbf{S}_1, \dots, \alpha_k \mathbf{S}_k$ . It means that after evaluating  $\mathcal{G}_{12\dots k}^{(k)}$  on the  $k$  arguments  $\alpha_1 \mathbf{S}_1, \dots, \alpha_k \mathbf{S}_k$  we get the  $k$ -electron detection probability density  $P_{12\dots k}^{(k)}$ . The probability of detecting  $k$  electrons at the points  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_k$  during the time intervals  $\langle t_1, t_1 + dt_1 \rangle, \dots, \langle t_k, t_k + dt_k \rangle$  will be then equal to  $P_{12\dots k}^{(k)} dt_1 \cdots dt_k$ .

In evaluating the expression (3.9) even for  $k = 2$  and simple electron states I have met big difficulties and to obtain reasonable results it has been necessary to make certain approximations, especially about the quasi-monochromaticity of the electrons. Fortunately,

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<sup>4</sup>for example,  $:\hat{\psi}_1^\dagger \hat{\psi}_2 \hat{\psi}_3^\dagger := -\hat{\psi}_1^\dagger \hat{\psi}_3^\dagger \hat{\psi}_2$ ; the minus sign appears due to antisymmetrical properties of fermion operators



the  $k$ -electron detection probability density  $P_{12\dots k}^{(k)}$  is proportional to the correlation function  $G_{12\dots k}^{(k)}$  with a good accuracy for quasi-monochromatic electrons. As we always deal with such electrons in practical situations that are to be described<sup>5</sup>, we will concentrate on the correlation functions according to Eq. (3.4) rather than the detection correlation functions such as Eq. (3.10).

The probability density of detecting  $k$  electrons at the points  $\mathbf{r}_1, \dots, \mathbf{r}_k$  and times  $t_1, \dots, t_k$ , can be obtained from the correlation function  $G_{12\dots k}^{(k)}$  by multiplication with the factor  $\prod_{i=1}^k v\alpha_i S_i$ , where  $v$  is the average velocity of the electrons.

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<sup>5</sup>We say that that electrons are quasi-monochromatic if their energy bandwidth  $\Delta E$  is much less than their mean kinetic energy  $E$ . For typical field-emission electrons this assumption is fully satisfied because  $\Delta E \approx 0.3$  eV, while  $E$  reaches the value of several keV.

# Chapter 4

## Field operators

As the field operators play the key role in the calculation of correlation functions, it will be useful to remind some of their basic properties in relation to the time evolution and the transformational properties between different creation and annihilation operators.

### 4.1 Heisenberg representation

We will use the Heisenberg representation of operators and states throughout this dissertation because it is more suitable for investigating correlations than the Schrödinger representation. Indeed, dealing with probabilities of events occurring at different moments of time (e.g. a multiple electron detection), it is almost impossible to use the Schrödinger representation in which there is only one time parameter expressing the evolution of the electron state.

As known, in the Heisenberg representation the states remain unchanged in time and the time dependence of physical variables is reflected by the change of the corresponding operators. Therefore the operators of physical variables are time-dependent, which is often expressed explicitly by writing the time argument  $t$ . The time evolution of any operator  $\hat{O}$  in the Heisenberg representation is given by the equation

$$\hat{O}(t') = \hat{U}^\dagger(t', t) \hat{O}(t) \hat{U}(t', t) \quad (4.1)$$

where  $\hat{U}(t)$  denotes the evolution operator of the system. This evolution operator is determined by the differential equation

$$i\hbar \frac{\partial}{\partial t'} \hat{U}(t', t) = \hat{H} \hat{U}(t', t) \quad (4.2)$$

with the initial condition that  $\hat{U}(t, t)$  is the unity operator. If the Hamiltonian  $\hat{H}$  of the system is not time-dependent, then it holds  $\hat{U}(t', t) = \hat{U}(t' - t) = \exp\left[-\frac{i}{\hbar} \hat{H}(t' - t)\right]$ .

Let  $\hat{\mathbf{r}}(t)$  denote the electron coordinate operator at the time  $t$  and  $|\mathbf{r}, t\rangle$  the eigenstate of this operator. So,  $|\mathbf{r}, t\rangle$  is the coordinate eigenstate at the moment  $t$ . However, at a different moment  $t' \neq t$  it is no more a coordinate eigenstate because the coordinate

operator  $\hat{\mathbf{r}}(t')$  has changed with respect to  $\hat{\mathbf{r}}(t)$  by the Heisenberg operator evolution. Thus the time argument  $t$  of the state  $|\mathbf{r}, t\rangle$  reflects the fact that  $|\mathbf{r}, t\rangle$  is the coordinate eigenstate at the moment  $t$  only and does not express any time evolution of the state  $|\mathbf{r}, t\rangle$ . So, it holds that

$$\hat{\mathbf{r}}(t)|\mathbf{r}, t\rangle = \mathbf{r}|\mathbf{r}, t\rangle, \quad \hat{\mathbf{r}}(t')|\mathbf{r}, t'\rangle = \mathbf{r}|\mathbf{r}, t'\rangle. \quad (4.3)$$

At the same time, the Heisenberg time evolution of the coordinate operator  $\hat{\mathbf{r}}$  is according to Eq. (4.1) given by

$$\hat{\mathbf{r}}(t') = \hat{U}^\dagger(t', t) \hat{\mathbf{r}}(t) \hat{U}(t', t). \quad (4.4)$$

Substituting Eq. (4.4) into Eq. (4.3b) and multiplying both sides with the operator  $\hat{U}(t', t) = [\hat{U}^\dagger(t', t)]^{-1}$  from the left, we get

$$\hat{\mathbf{r}}(t) \hat{U}(t', t)|\mathbf{r}, t'\rangle = \mathbf{r} \hat{U}(t', t)|\mathbf{r}, t'\rangle, \quad (4.5)$$

from which, together with Eq. (4.3a) it follows

$$|\mathbf{r}, t\rangle = \hat{U}(t', t)|\mathbf{r}, t'\rangle. \quad (4.6)$$

This equation shows the relation between the states  $|\mathbf{r}, t\rangle$  at different times.

The electron field operators are simply the operators that annihilate and create an electron at the state  $|\mathbf{r}, t\rangle$  (or, in other words, annihilate and create an electron at the space-time point  $(\mathbf{r}, t)$ ):

$$\hat{\psi}^\dagger(\mathbf{r}, t)|\text{vac}\rangle = |\mathbf{r}, t\rangle, \quad \hat{\psi}(\mathbf{r}, t)|\mathbf{r}', t\rangle = \delta^3(\mathbf{r} - \mathbf{r}')|\text{vac}\rangle, \quad (4.7)$$

and they are, as any other operators, also governed by the equation (4.1):

$$\hat{\psi}^\dagger(\mathbf{r}, t') = \hat{U}^\dagger(t', t) \hat{\psi}^\dagger(\mathbf{r}, t) \hat{U}(t', t), \quad \hat{\psi}(\mathbf{r}, t') = \hat{U}^\dagger(t', t) \hat{\psi}(\mathbf{r}, t) \hat{U}(t', t). \quad (4.8)$$

Here  $\delta^3(\mathbf{r} - \mathbf{r}')$  denotes the three-dimensional  $\delta$ -function and  $|\text{vac}\rangle$  is the vacuum ket. The field operators satisfy the fermion anticommutation relations

$$\begin{aligned} \hat{\psi}^\dagger(\mathbf{r}, t) \hat{\psi}(\mathbf{r}', t) + \hat{\psi}(\mathbf{r}', t) \hat{\psi}^\dagger(\mathbf{r}, t) &= \delta^3(\mathbf{r} - \mathbf{r}'), \\ \hat{\psi}(\mathbf{r}, t) \hat{\psi}(\mathbf{r}', t) + \hat{\psi}(\mathbf{r}', t) \hat{\psi}(\mathbf{r}, t) &= 0, \\ \hat{\psi}^\dagger(\mathbf{r}, t) \hat{\psi}^\dagger(\mathbf{r}', t) + \hat{\psi}^\dagger(\mathbf{r}', t) \hat{\psi}^\dagger(\mathbf{r}, t) &= 0. \end{aligned} \quad (4.9)$$

## 4.2 Relations between creation and annihilation operators of different states

It will be very useful for the following calculations to express the field operators  $\hat{\psi}(\mathbf{r}, t)$  and  $\hat{\psi}^\dagger(\mathbf{r}, t)$  in terms of the creation and annihilation operators corresponding to other set of

states than the coordinate eigenstates. So, let  $\{|p\rangle\}$  be a complete orthonormal set of one-electron states indexed by the variable  $p$  (in the following we will call these states generally “modes”). An example can be the set of the momentum eigenstates. Using the expansion of the unity operator, we can write

$$|\mathbf{r}, t\rangle = \sum_p |p\rangle \langle p|\mathbf{r}, t\rangle = \sum_p K^*(\mathbf{r}, t|p) |p\rangle, \quad (4.10)$$

where  $K(\mathbf{r}, t|p) = \langle \mathbf{r}, t|p\rangle$  is the probability amplitude that an electron gets to state  $|\mathbf{r}, t\rangle$  from the state  $|p\rangle$ , or, in other words, the one-electron propagator in the  $p$ - $\mathbf{r}$  representation. If  $\hat{a}^\dagger(p)$  is the creation operator creating an electron in the state  $|p\rangle$  such that  $\hat{a}^\dagger(p)|\text{vac}\rangle = |p\rangle$ , we get by combining Eqs. (4.10) and (4.7) simply

$$\hat{\psi}^\dagger(\mathbf{r}, t) = \sum_p K^*(\mathbf{r}, t|p) \hat{a}^\dagger(p) \quad (4.11)$$

and by the Hermite conjugation

$$\hat{\psi}(\mathbf{r}, t) = \sum_p K(\mathbf{r}, t|p) \hat{a}(p). \quad (4.12)$$

Thus the relations (4.11) and (4.12) connect the creation (or annihilation) operators of different states. The following anticommutation relations hold for the operators  $\hat{a}^\dagger(p)$  and  $\hat{a}(p)$ :

$$\begin{aligned} \hat{a}^\dagger(p)\hat{a}(p') + \hat{a}(p')\hat{a}^\dagger(p) &= \delta(p, p'), \\ \hat{a}(p)\hat{a}(p') + \hat{a}(p')\hat{a}(p) &= 0, \\ \hat{a}^\dagger(p)\hat{a}^\dagger(p') + \hat{a}^\dagger(p')\hat{a}^\dagger(p) &= 0, \end{aligned} \quad (4.13)$$

where  $\delta(p, p') = \delta_{pp'}$  denotes the Kronecker delta.

Usually plane waves are considered as the modes of the field. As we have not made any such assumption about the modes  $p$ , our formulas (4.11) and (4.12) are more general and can be used for describing electron fields also in external electromagnetic fields.

# Chapter 5

## Electron state

If we want to describe coherence properties of electrons emitted by a specific source, it is of fundamental importance to know what a multi-particle state is produced by the source. We will concentrate on the most coherent electron source known up to now, a field-emission gun. Unfortunately, it is not good known what a state is produced by such a source because the mechanism of tunneling from metal to vacuum is very complicated. For example, up to now there is no theoretical explanation for the observed low- and high-energy tails observed in the field-emission energy spectra [18]. It might therefore seem hopeless to calculate the correlation functions if we even do not know exactly what state we deal with. Fortunately, in spite of the uncertainty of the state emitted by the field-emission tip, it is possible to make some estimations of the form of this state.

The emitted electrons originate from the metal (a tungsten monocrystal at a room temperature), where they are in a quasi-equilibrium state that is very close to a thermal state. We say “close” because the equilibrium is partially broken by the applied electrostatic field. In a thermal equilibrium state the density operator of the field has the form

$$\hat{\rho}_{\text{thermal}} = \frac{\exp(-\beta\hat{H})}{\text{Tr}\{\exp(-\beta\hat{H})\}} \quad (5.1)$$

and is therefore diagonal in the energy representation. This means that there is no mutual coherence or correlation between parts of the field with different energies. It does not seem to be likely that an additional coherence would come into existence during the tunneling process, even if we cannot, of course, exclude this possibility totally. This suggests that the density operator of the emitted field is diagonal in the representation of states that come out from the original energy eigenstates in the metal by the tunneling process. We will call these states “modes” in the following and, to avoid problems with the normalization of the density operator, we will suppose that the set of the modes is discrete.

The best approximation for the real electron state produced by the field-emission source seems to be the so-called chaotic state that is described in the following. Therefore we will concentrate especially on this state in this dissertation. However, we will give also some other, maybe less physical examples of possible electron states to show the properties of the electron correlations in a more complex way.

## 5.1 Chaotic state

The chaotic state was introduced first by Glauber [19] in optics and has two basic properties: first, each mode of the chaotic field has a maximum entropy provided that the mean number of photons in this mode is given, and second, the density operator of the field can be expressed as a direct product of the one-mode density operators. The second property means simply that in the chaotic state the individual modes of the field are totally uncorrelated.

If we now generalize this definition on the case of electrons, the situation is very simple because there are only two possibilities of occupation of each mode  $p$  — either there is one electron or there is none. If we denote the corresponding kets as  $|1\rangle_p$  and  $|0\rangle_p$ , respectively, and the mean number of electrons in this mode as  $n(p)$ , respectively, the most general form of the density operator of the mode  $p$  can be written in the basis  $\{|1\rangle_p, |0\rangle_p\}$  as

$$\hat{\rho}(p) = \begin{pmatrix} n(p) & a \\ a^* & 1 - n(p) \end{pmatrix}, \quad (5.2)$$

where  $a$  is a complex number. The maximum entropy condition then yields  $a = 0$  and therefore

$$\hat{\rho}(p) = \begin{pmatrix} n(p) & 0 \\ 0 & 1 - n(p) \end{pmatrix} = n(p) |1\rangle_p \langle 1|_p + [1 - n(p)] |0\rangle_p \langle 0|_p. \quad (5.3)$$

With respect to the second property of the chaotic state, the total density operator is

$$\hat{\rho} = \prod_p \hat{\rho}(p) = \prod_p \{[1 - n(p)] |0\rangle_p \langle 0|_p + n(p) |1\rangle_p \langle 1|_p\}. \quad (5.4)$$

The product is made here over all, also the non-occupied modes of the field. If some mode  $p$  is not occupied, it holds  $n(p) = 0$  and its density operator corresponds just to the vacuum state. The total mean number  $N$  of electrons in the system is given by the sum of the average numbers of electrons in all modes, that is,  $N = \sum_p n(p)$ . The probability that any electron will be found in the mode  $p$  is then of course  $f(p) = n(p)/N$ .

## 5.2 Generalized chaotic state

Another example of a state satisfying the condition of incoherence between individual modes of the electron field is a slight generalization of the state introduced by Silverman [15]. In this state the density operator describing the electron field has the form

$$\hat{\rho} = \sum_{n=0}^{\infty} P(n) \hat{\rho}_n, \quad (5.5)$$

where

$$\hat{\rho}_n = \sum_{\substack{p_1, \dots, p_n \\ \text{all } p_i \text{ different}}} f_n(p_1, p_2, \dots, p_n) |p_1, p_2, \dots, p_n\rangle \langle p_1, p_2, \dots, p_n|. \quad (5.6)$$

Here  $P(n)$  is the normalized distribution of particle number,  $f_n(p_1, p_2, \dots, p_n)$  is the probability that if there are  $n$  electrons in the system, they occupy the modes  $p_1, \dots, p_n$ , and the ket  $|p_1, \dots, p_n\rangle$  denotes the totally antisymmetrical Fock state of  $n$  electrons in modes  $p_1, \dots, p_n$ . The reason why there are no terms in the sum with some  $p_i, p_j$  equal is that in this case the state  $|p_1, \dots, p_n\rangle$  does not have any physical meaning due to the Pauli principle. The functions  $f_n(p_1, p_2, \dots, p_n)$  are symmetrical in all variables and normalized as follows:

$$\sum_{\substack{p_1, \dots, p_n \\ \text{all } p_i \text{ different}}} f_n(p_1, p_2, \dots, p_n) = 1. \quad (5.7)$$

In the next chapter we will see that the state (5.5) is a special case of the chaotic state.

### 5.3 Pure Fock state of $n$ electrons in modes $p_1, \dots, p_n$

An interesting state of the electron field is a pure Fock state of  $n$  electrons that occupy the modes  $p_1, p_2, \dots, p_n$ . The density operator has now a very simple form that can be written as

$$\hat{\rho} = |p_1, p_2, \dots, p_n\rangle\langle p_1, p_2, \dots, p_n|. \quad (5.8)$$

### 5.4 Wave-packet state

The last example of a possible state of the electron field is the state of  $n$  equal wave packets that have been randomly emitted from a source. Unfortunately, I have not managed yet to calculate the correlation function of an arbitrary order for this state, but only the  $n$ -electron one (i.e., the correlation function of the same order as is the number of electrons). Moreover, it is completely unclear how to perform the averaging over the possible emission times of the packets in a mathematically pure way. Therefore I will just refer at this place to the article by Toyoshima and Endo [20] where the two-electron correlation function for two wave packets is calculated.

# Chapter 6

## Calculation of correlation functions

Now we are coming to a very important part of this dissertation, the calculation of the correlation functions for the individual states introduced in Chapter 5. The fundamental part of this chapter will be devoted to the calculation of the correlation function for the chaotic state and the obtained results will be then used often in the following chapters because I think that this state describes the field-emission beams in the best way. The correlation functions for the other two states (the generalized chaotic state and the pure state) will be calculated just to cover a larger spectrum of the possible states and the results will not be used any more. Before proceeding in the calculation, it is necessary to make several assumptions.

### 6.1 Assumptions for the calculation

For practical calculations of the correlation functions, it will be very convenient to make two important assumptions that are fully acceptable for the case of the field-emission electrons.

First, we will assume that the kinetic energy of the electrons is much less than their rest energy, which will enable us to use the non-relativistic quantum theory for the calculations. This is appropriate because in a typical electron correlation experiment we deal with kinetic energies of order of kiloelectronvolts (in Tübingen it is 2 keV), which is much less than the electron rest energy of 512 keV. Moreover, I expect that the results obtained by the relativistic theory even for higher energies would not differ from our ones very much. The reason is that the principle of fermion indistinguishableness that in fact causes all the correlations considered in this dissertation holds both in the relativistic and non-relativistic theories.

Second, we will assume that the Coulomb interaction between the electrons can be neglected. Indeed, the average distance of the electrons in a beam of an intensity typical for field-emission beams is so large that the Coulomb force does not have any effect on the electron velocity and position.

Beside these two important assumptions, we will consider in this chapter the case of spin-polarized electrons. This will enable us to neglect the spinor structure of the electron field and to treat it as a scalar field. However, this does not mean at all that we neglect



the spin completely and in Chapter 7 we will calculate the correlation functions also for partially polarized or completely unpolarized electrons.

## 6.2 Correlation function for the chaotic state

### 6.2.1 An alternative form of the chaotic state

As we have seen, the density operator of the chaotic state is

$$\hat{\rho} = \prod_p \hat{\rho}(p) = \prod_p \{ [1 - n(p)] |0\rangle_p \langle 0|_p + n(p) |1\rangle_p \langle 1|_p \}. \quad (6.1)$$

Now, it will be useful to express this operator in a form that is more convenient for the calculation and also shows the structure of the chaotic state in a way alternative to Eq. (6.1). To do so, we expand the product in Eq. (6.1). As a result, we get terms with various particle numbers  $n$  with  $n$  going from zero to infinity. To obtain the zero-particle (or vacuum) part of  $\hat{\rho}$ , we have to combine the states  $|0\rangle_p \langle 0|_p$  for all  $p$  in Eq. (6.1). The coefficient of the vacuum state in the density operator (6.1) will therefore be the product of the coefficients of the states  $|0\rangle_p \langle 0|_p$ , i.e., of the terms  $[1 - n(p)]$ . Similarly, the coefficient of the state with a momentum  $p$  in will be  $n(p)$  times a product of the terms  $[1 - n(p')]$  for all  $p' \neq p$ . Thus the zero- and one-particle parts  $\hat{R}_0, \hat{R}_1$  of the density operator  $\hat{\rho}$  are

$$\hat{R}_0 = |\text{vac}\rangle \langle \text{vac}| \prod_p [1 - n(p)], \quad \hat{R}_1 = \sum_{p_1} n(p_1) |p_1\rangle \langle p_1| \prod_{\text{all } p \text{ without } p_1} [1 - n(p)]. \quad (6.2)$$

In an analogous way we can write the  $n$ -particle part  $\hat{R}_n$  of  $\hat{\rho}$ :

$$\hat{R}_n = \frac{1}{n!} \sum_{\substack{p_1, \dots, p_n \\ \text{all } p_i \text{ different}}} n(p_1) \cdots n(p_n) |p_1, \dots, p_n\rangle \langle p_1, \dots, p_n| \prod_{\text{all } p \text{ without } p_1, \dots, p_n} [1 - n(p)]. \quad (6.3)$$

The summation here is made over all the combinations of  $p_1, \dots, p_n$  with all  $p_i$  different from each other. To simplify the equations, we will denote such a sum as  $\sum'_{p_1, \dots, p_n}$  in the following. As each set  $\{p_1, \dots, p_n\}$  appears  $n!$  times in the sum (6.3) while in Eq. (6.1) only once, we had to divide the sum by  $n!$  to obtain the correct result. Now, to write  $\hat{R}_n$  in a more compact way, we denote  $S = \prod_p [1 - n(p)]$  and  $h(p) = n(p)/[1 - n(p)]$ . Then

$$\hat{R}_n = \frac{S}{n!} \sum'_{p_1, \dots, p_n} h(p_1) \cdots h(p_n) |p_1, \dots, p_n\rangle \langle p_1, \dots, p_n|. \quad (6.4)$$

It will be very useful to express the density operator  $\hat{\rho}$  as an incoherent superposition of  $n$ -electron density operators  $\hat{\rho}_n$  in the following way:

$$\hat{\rho} = \sum_{n=0}^{\infty} P_n \hat{\rho}_n, \quad (6.5)$$

where

$$\hat{\rho}_n = \frac{\hat{R}_n}{P_n}, \quad P_n = \text{Tr}(\hat{R}_n). \quad (6.6)$$

This ensures that the operators  $\hat{\rho}_n$  are properly normalized. The trace of the operator  $\hat{R}_n$  expresses the probability that there are  $n$  electrons in the system. With the help of Eq. (6.4) it can be simplified as follows:

$$P_n = \text{Tr}(\hat{R}_n) = \frac{S}{n!} \sum_{p_1, \dots, p_n} ' h(p_1) \cdots h(p_n) \text{Tr}|p_1, \dots, p_n\rangle\langle p_1, \dots, p_n| = \frac{S}{n!} \sum_{p_1, \dots, p_n} ' h(p_1) \cdots h(p_n). \quad (6.7)$$

Now, it will be very useful to assume that the number  $K$  of states  $|p\rangle$  contributing significantly to the density operator  $\hat{\rho}$  is very large, as in the case of a quasi-continuous momentum spectrum. We can do that because, in fact, the only reason why the discrete set of the modes has been introduced was to avoid problems with the normalization of the density operators. However, from the physical point of view, there is no reason for the set of the modes to be discrete. This means that even if the spectrum of the modes is very narrow (corresponding to almost monochromatic electrons), there are many occupied modes. On the other hand, if  $K$  is very large, it holds

$$\sum_{p_1, \dots, p_n} ' h(p_1) \cdots h(p_n) \approx \sum_{p_1, \dots, p_n} h(p_1) \cdots h(p_n). \quad (6.8)$$

Indeed, the relative number of combinations  $p_1, \dots, p_n$  with  $p_i = p_j$  for some  $i, j$  ( $i \neq j$ ) with respect to the number of all combinations<sup>1</sup> behaves like  $1/K$  and hence goes to zero for large  $K$ . Therefore also the normalized difference

$$\frac{\sum_{p_1, \dots, p_n} h(p_1)h(p_2) \cdots h(p_n) - \sum_{p_1, \dots, p_n} ' h(p_1)h(p_2) \cdots h(p_n)}{\sum_{p_1, \dots, p_n} h(p_1)h(p_2) \cdots h(p_n)}$$

goes to zero and the approximation (6.8) is thus justified. We point out that Eq. (6.8) turns into identity when we go over to the continuous spectrum because  $K$  then goes to infinity. Then Eq. (6.8) holds exactly and is in fact no approximation.

From the assumption that  $K$  is large it follows that the values  $n(p)$  are very small. Then we can calculate the product  $S$  easily using the properties of the exponential and logarithm functions:

$$S = \prod_p [1 - n(p)] = \exp \left\{ \sum_p \ln[1 - n(p)] \right\} \approx \exp \left[ - \sum_p n(p) \right] = e^{-N}. \quad (6.9)$$

Substituting Eq. (6.8) into Eq. (6.7), we get for  $P_n$

$$P_n \approx \frac{S}{n!} \sum_{p_1, \dots, p_n} h(p_1) \cdots h(p_n) = \frac{e^{-N} \mathcal{N}^n}{n!}, \quad (6.10)$$

---

<sup>1</sup>of course, we consider now the momenta belonging to the set of the corresponding  $K$  eigenvalues only

where  $\mathcal{N} = \sum_p h(p)$ . To see what is  $\mathcal{N}$  equal to, we expand  $h(p)$  using  $n(p)$ :

$$h(p) = \frac{n(p)}{1 - n(p)} = n(p) + [n(p)]^2 + \dots \quad (6.11)$$

As the numbers  $n(p)$  are small, we can neglect now all the higher terms with respect to  $n(p)$  and hence we see that  $h(p) \approx n(p)$  and  $\mathcal{N} \approx N$ . The electron-number distribution  $P_n$  thus becomes a Poissonian distribution

$$P_n = \frac{e^{-N} N^n}{n!}. \quad (6.12)$$

With the help of Eqs. (6.6), (6.9) and (6.4), the  $n$ -electron density operator  $\hat{\rho}_n$  can be expressed as

$$\hat{\rho}_n = \frac{\hat{R}_n}{P_n} = \frac{1}{N^n} \sum'_{p_1, \dots, p_n} h(p_1)h(p_2) \cdots h(p_n) |p_1, \dots, p_n\rangle \langle p_1, \dots, p_n|. \quad (6.13)$$

Even if we could write now again the simple sum instead of  $\sum'_{p_1, \dots, p_n}$ , we will not do so because this form will be more useful for further calculations. Defining  $f(p) = n(p)/N$  and approximating  $h(p)$  by  $n(p)$ , Eq. (6.13) becomes finally

$$\hat{\rho}_n = \sum'_{p_1, \dots, p_n} f(p_1)f(p_2) \cdots f(p_n) |p_1, \dots, p_n\rangle \langle p_1, \dots, p_n|. \quad (6.14)$$

The function  $f(p)$  expresses simply the probability that an electron randomly chosen from the system will be found in the mode  $p$ . Thus  $f(p)$  can be called the one-electron mode-spectrum distribution (in analogy to e.g. one-electron energy distribution). The mode-spectrum distribution satisfies the normalization condition

$$\sum_p f(p) = 1. \quad (6.15)$$

Now we are ready to go over to the derivation of the correlation function.

### 6.2.2 Calculation of the $k$ -electron correlation function for the chaotic state

The  $k$ -electron correlation function expressing the probability density of finding  $k$  electrons at points  $\mathbf{r}_1, \dots, \mathbf{r}_k$  and times  $t_1, \dots, t_k$  was defined in Chapter (3) as

$$G^{(k)} = \text{Tr} \left\{ \hat{\rho} \hat{\psi}_1^\dagger \cdots \hat{\psi}_k^\dagger \hat{\psi}_k \cdots \hat{\psi}_1 \right\}. \quad (6.16)$$

We have omitted now even the indexes  $1, 2, \dots, k$  to save place for another index that is about to come. With the help of the relations (4.11) and (4.12) we can express the product of the field operators in Eq. (6.16) in the following way:

$$\hat{\psi}_1^\dagger \cdots \hat{\psi}_k^\dagger \hat{\psi}_k \cdots \hat{\psi}_1 = \sum_{\{q\}, \{q'\}} K_1^*(q_1) \cdots K_k^*(q_k) K_k(q'_k) \cdots K_1(q'_1) \hat{a}^\dagger(q_1) \cdots \hat{a}^\dagger(q_k) \hat{a}(q'_k) \cdots \hat{a}(q'_1). \quad (6.17)$$

Here  $\sum_{\{q\},\{q'\}}$  expresses the summation over  $q_1, \dots, q_k$  and  $q'_1, \dots, q'_k$ . Using the form (6.5) of the density operator  $\hat{\rho}$ , we can write

$$G^{(k)} = \sum_{n=0}^{\infty} P_n \text{Tr} \left\{ \hat{\rho}_n \hat{\psi}_1^\dagger \cdots \hat{\psi}_k^\dagger \hat{\psi}_k \cdots \hat{\psi}_1 \right\} = \sum_{n=0}^{\infty} P_n G_n^{(k)}, \quad (6.18)$$

where  $G_n^{(k)}$  is the correlation function corresponding to the  $n$ -electron density operator  $\hat{\rho}_n$ . With the help of Eqs. (6.14) and (6.17), this correlation function can be now expressed as

$$G_n^{(k)} = \sum_{\{q\},\{q'\}} \sum'_{p_1, \dots, p_n} f(p_1) \cdots f(p_n) K_1^*(q_1) \cdots K_k^*(q_k) K_k(q'_k) \cdots K_1(q'_1) \\ \times \text{Tr} \left\{ |p_1, \dots, p_n\rangle \langle p_1, \dots, p_n| \hat{a}^\dagger(q_1) \cdots \hat{a}^\dagger(q_k) \hat{a}(q'_k) \cdots \hat{a}(q'_1) \right\}. \quad (6.19)$$

Now, the key part of the calculation comes. As is shown in Appendix A, page 33, it follows from the anticommutation relations (4.13) that the trace in Eq. (6.19) is equal to

$$\text{Tr} \left\{ |p_1, \dots, p_n\rangle \langle p_1, \dots, p_n| \hat{a}^\dagger(q_1) \cdots \hat{a}^\dagger(q_k) \hat{a}(q'_k) \cdots \hat{a}(q'_1) \right\} \\ = \sum_{P \in \mathcal{P}} \text{sign}(P) \delta(q_1, q'_{P(1)}) \cdots \delta(q_k, q'_{P(k)}) \sum_{\substack{i_1, \dots, i_k=1 \\ \text{all } i \text{ different}}}^n \delta(p_{i_1}, q_1) \cdots \delta(p_{i_k}, q_k). \quad (6.20)$$

Here  $P$ ,  $P^{(j)}$ , and  $\text{sign}(P)$  denote a permutation of  $k$  indexes  $1, 2, \dots, k$ , and the  $j$ -th element and sign of this permutation, respectively. The first sum is made over the set  $\mathcal{P}$  of all permutations. The fermion nature of the electrons is demonstrated by the factor  $\text{sign}(P)$ .

We make our notation more clear on an example with  $k = 3$ . Then the set of all permutations is  $\mathcal{P} = \{(1, 2, 3), (3, 1, 2), (2, 3, 1), (1, 3, 2), (3, 2, 1), (2, 1, 3)\}$  and if we choose the permutation  $P = (2, 1, 3)$ , we have  $P^{(1)} = 2$ ,  $P^{(2)} = 1$ ,  $P^{(3)} = 3$ , and  $\text{sign}(P) = -1$  because the permutation is odd.

Substituting Eq. (6.20) to Eq. (6.19) we obtain

$$G_n^{(k)} = \sum_{\{q\},\{q'\}} K_1^*(q_1) \cdots K_k^*(q_k) K_k(q'_k) \cdots K_1(q'_1) \\ \sum_{P \in \mathcal{P}} \text{sign}(P) \delta(q_1, q'_{P(1)}) \cdots \delta(q_k, q'_{P(k)}) \\ \sum_{\substack{i_1, \dots, i_k=1 \\ \text{all } i \text{ different}}}^n \sum'_{p_1, \dots, p_n} f(p_1) \cdots f(p_n) \delta(p_{i_1}, q_1) \cdots \delta(p_{i_k}, q_k). \quad (6.21)$$

In Eq. (6.21) we calculate first the the sum over the  $n$  momenta  $p_1, \dots, p_n$ . It is useful to note that for every possible combination  $i_1, \dots, i_k$  with all  $i$  different it holds due to the normalization condition (6.15) of the function  $f(p)$

$$\sum'_{p_1, \dots, p_n} f(p_1) \cdots f(p_n) \delta(p_{i_1}, q_1) \cdots \delta(p_{i_k}, q_k) = f(q_1) \cdots f(q_k), \quad (6.22)$$

that is, the sum does not depend on the indexes  $i_1, \dots, i_k$ . We have used here again the approximation  $\sum' \rightarrow \sum$ . Therefore all the  $n(n-1)\cdots(n-k+1)$  terms (we will denote this product as  $V(k, n)$  in the following) in the sum over  $i_1, \dots, i_k$  all different from each other give the same result and we obtain

$$\sum_{\substack{i_1, \dots, i_k=1 \\ \text{all } i \text{ different}}}^n \sum_{p_1, \dots, p_n}' f(p_1) \cdots f(p_n) \delta(p_{i_1}, q_1) \cdots \delta(p_{i_k}, q_k) = V(k, n) f(q_1) \cdots f(q_k). \quad (6.23)$$

Next we substitute Eq. (6.23) into Eq. (6.21) and evaluate the sum over the momenta  $q'_1, \dots, q'_k$ :

$$\begin{aligned} G_n^{(k)} &= V(k, n) \sum_{\{q\}, \{q'\}} f(q_1) \cdots f(q_k) K_1^*(q_1) \cdots K_k^*(q_k) K_k(q'_k) \cdots K_1(q'_1) \\ &\quad \sum_{P \in \mathcal{P}} \text{sign}(P) \delta(q_1, q'_{P(1)}) \cdots \delta(q_k, q'_{P(k)}) \\ &= V(k, n) \sum_{P \in \mathcal{P}} \text{sign}(P) \sum_{q_1, \dots, q_k} f(q_1) \cdots f(q_k) K_1^*(q_1) \cdots K_k^*(q_k) K_{P(1)}(q_1) \cdots K_{P(k)}(q_k). \end{aligned} \quad (6.24)$$

Gathering the terms with equal  $q_i$ , we can write the result in the form

$$G_n^{(k)} = V(k, n) \sum_{P \in \mathcal{P}} \text{sign}(P) \sum_q f(q) K_1^*(q) K_{P(1)}(q) \cdots \sum_q f(q) K_k^*(q) K_{P(k)}(q), \quad (6.25)$$

We see that the correlation functions for different electron numbers differ from each other only by the multiplicative factor  $V(k, n)$ . Therefore due to Eq. (6.18) the evaluation of the “total” correlation function  $G^{(k)}$  reduces now to averaging this factor for the Poissonian distribution (6.12). It is easy to verify that for this distribution  $\langle V(k, n) \rangle = \langle n(n-1)\cdots(n-k+1) \rangle = N^k$  holds and for  $G^{(k)}$  we get

$$G^{(k)} = \sum_{n=0}^{\infty} P_n G_n^{(k)} = N^k \sum_{P \in \mathcal{P}} \text{sign}(P) \sum_q f(q) K_1^*(q) K_{P(1)}(q) \cdots \sum_q f(q) K_k^*(q) K_{P(k)}(q). \quad (6.26)$$

If we denote

$$\Gamma_{ij} = N \sum_q f(q) K_i^*(q) K_j(q), \quad (6.27)$$

the equation (6.26) can be written in the form of a determinant

$$G^{(k)} = \begin{vmatrix} \Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1k} \\ \Gamma_{21} & \Gamma_{22} & \cdots & \Gamma_{2k} \\ \vdots & \vdots & & \vdots \\ \Gamma_{k1} & \Gamma_{k2} & \cdots & \Gamma_{kk} \end{vmatrix}. \quad (6.28)$$

This result has been obtained also by Saito *et al.* [16] and the whole calculation has been published in [21].

It is very useful to note that  $\Gamma_{ij} = \langle \hat{\psi}_i^\dagger \hat{\psi}_j \rangle = \text{Tr}\{\hat{\rho} \hat{\psi}_i^\dagger \hat{\psi}_j\}$  (for the proof see Appendix B, page 35). Thus the elements of the determinant in Eq. (6.28) are in fact the correlation functions of the first order with unequal arguments (the field operators correspond to two different space time points) referred to as the cross-correlation functions in quantum optics (see [1], pages 422 and 583—585). We also introduce the complex degree of coherence

$$\gamma_{ij} = \frac{\Gamma_{ij}}{\sqrt{\Gamma_{ii}\Gamma_{jj}}}, \quad (6.29)$$

which is the normalized cross-correlation function. To express the correlation function  $G^{(k)}$  in terms of the complex degrees of coherence, we use the properties of determinants and the fact that  $\Gamma_{ii} = G_i^{(1)}$  for all  $i$ , which follows from Eq. (6.28) for  $k = 1$ . Then

$$G_{12\dots k}^{(k)} = G_1^{(1)} G_2^{(1)} \dots G_k^{(1)} \begin{vmatrix} 1 & \gamma_{12} & \dots & \gamma_{1k} \\ \gamma_{21} & 1 & \dots & \gamma_{2k} \\ \vdots & \vdots & & \vdots \\ \gamma_{k1} & \gamma_{k2} & \dots & 1 \end{vmatrix}. \quad (6.30)$$

### 6.3 Correlation function for the generalized chaotic state

Comparing Eqs. (5.5) and (5.6) with Eqs. (6.5) and (6.14), we see that the state (5.5) is a special case of the chaotic state. The calculation of the correlation function is completely analogous to the one performed for the chaotic state. Therefore we will show here only the steps where there is a difference.

We express the correlation function again as the weighed average of the correlation functions corresponding to  $n$ -electron density operators in analogy with Eq. (6.18):

$$G^{(k)} = \sum_{n=0}^{\infty} P(n) \text{Tr}\{\hat{\rho}_n \hat{\psi}_1^\dagger \dots \hat{\psi}_k^\dagger \hat{\psi}_k \dots \hat{\psi}_1\} = \sum_{n=0}^{\infty} P(n) G_n^{(k)}. \quad (6.31)$$

For the correlation function  $G_n^{(k)}$  corresponding to the  $n$ -electron density operator  $\hat{\rho}_n$  we get in analogy to Eq. (6.21)

$$G_n^{(k)} = \sum_{\{q\}, \{q'\}} K_1^*(q_1) \dots K_k^*(q_k) K_k(q'_k) \dots K_1(q'_1) \sum_{P \in \mathcal{P}} \text{sign}(P) \delta(q_1, q'_{P(1)}) \dots \delta(q_k, q'_{P(k)}) \sum_{\substack{i_1, \dots, i_k=1 \\ \text{all } i \text{ different}}}^n \sum_{p_1, \dots, p_n}' f_n(p_1, \dots, p_n) \delta(p_{i_1}, q_1) \dots \delta(p_{i_k}, q_k). \quad (6.32)$$

Thanks to the symmetry of the function  $f_n(p_1, \dots, p_n)$  it holds

$$\begin{aligned} & \sum_{\substack{i_1, \dots, i_k=1 \\ \text{all } i \text{ different}}}^n \sum_{p_1, \dots, p_n}' f_n(p_1, \dots, p_n) \delta(p_{i_1}, q_1) \cdots \delta(p_{i_k}, q_k) \\ &= V(k, n) \sum_{q_{k+1}, \dots, q_n} f_n(q_1, \dots, q_n) = V(k, n) f_n^{(k)}(q_1, \dots, q_k). \end{aligned} \quad (6.33)$$

In this way, the  $k$ -electron momentum distribution  $f_n^{(k)}(q_1, \dots, q_k)$  corresponding to the  $n$ -electron density operator is defined. After the summation over  $\{q'\}$  and  $n$  we then get

$$\begin{aligned} G^{(k)} &= \sum_{P \in \mathcal{P}} \text{sign}(P) \sum_{q_1, \dots, q_k} K_1^*(q_1) \cdots K_k^*(q_k) K_{P(1)}(q_1) \cdots K_{P(k)}(q_k) \\ &\quad \times \sum_{n=k}^{\infty} P(n) V(k, n) f_n^{(k)}(q_1, \dots, q_k). \end{aligned} \quad (6.34)$$

If we define the  $k$ -electron momentum distribution  $f^{(k)}(q_1, \dots, q_k)$  corresponding to the total density operator  $\hat{\rho}$  as

$$f^{(k)}(q_1, \dots, q_k) = \frac{\sum_{n=k}^{\infty} P(n) V(k, n) f_n^{(k)}(q_1, \dots, q_k)}{\langle V \rangle^{(k)}} \quad (6.35)$$

where  $\langle V \rangle^{(k)} = \sum_{n=k}^{\infty} P(n) V(k, n)$ , we obtain the final form of the correlation function

$$G^{(k)} = \langle V \rangle^{(k)} \sum_{P \in \mathcal{P}} \text{sign}(P) \sum_{q_1, \dots, q_k} f^{(k)}(q_1, \dots, q_k) K_1^*(q_1) \cdots K_k^*(q_k) K_{P(1)}(q_1) \cdots K_{P(k)}(q_k). \quad (6.36)$$

## 6.4 Correlation function for the $n$ -electron pure state

Next we will calculate the  $k$ -electron correlation function for the  $n$ -electron pure state defined in Chap. (1),

$$\hat{\rho} = |p_1, p_2, \dots, p_n\rangle \langle p_1, p_2, \dots, p_n|. \quad (6.37)$$

The calculation will be again very similar to the previous cases. Combining Eqs. (6.37), (4.12), (4.11) and (3.4) we get

$$\begin{aligned} G^{(k)} &= \sum_{\{q\}, \{q'\}} K_1^*(q_1) \cdots K_k^*(q_k) K_k(q'_k) \cdots K_1(q'_1) \\ &\quad \times \text{Tr} \left\{ |p_1, \dots, p_n\rangle \langle p_1, \dots, p_n| \hat{a}^+(q_1) \cdots \hat{a}^+(q_k) \hat{a}(q'_k) \cdots \hat{a}(q'_1) \right\}. \end{aligned} \quad (6.38)$$

Note that there is no summation over the modes  $p_1, \dots, p_n$  because the information about the electron state is contained now in the set  $\{p_1, \dots, p_n\}$ . Using the expression (6.20) for the trace, we obtain

$$\begin{aligned} G^{(k)} &= \sum_{\{q\}, \{q'\}} K_1^*(q_1) \cdots K_k^*(q_k) K_k(q'_k) \cdots K_1(q'_1) \\ &\quad \sum_{P \in \mathcal{P}} \text{sign}(P) \delta(q_1, q'_{P(1)}) \cdots \delta(q_k, q'_{P(k)}) \sum_{\substack{i_1, \dots, i_k=1 \\ \text{all } i \text{ different}}}^n \delta(p_{i_1}, q_1) \cdots \delta(p_{i_k}, q_k). \end{aligned} \quad (6.39)$$

We perform first the summation over  $q_1, \dots, q_k$  and then over  $q'_1, \dots, q'_k$ :

$$\begin{aligned}
G^{(k)} &= \sum_{\{q'\}} \sum_{\substack{i_1, \dots, i_k=1 \\ \text{all } i \text{ different}}}^n K_1^*(p_{i_1}) \cdots K_k^*(p_{i_k}) K_k(q'_k) \cdots K_1(q'_1) \\
&\quad \times \sum_{P \in \mathcal{P}} \text{sign}(P) \delta(p_{i_1}, q'_{P(1)}) \cdots \delta(p_{i_k}, q'_{P(k)}) \\
&= \sum_{\substack{i_1, \dots, i_k=1 \\ \text{all } i \text{ different}}}^n \sum_{P \in \mathcal{P}} \text{sign}(P) K_1^*(p_{i_1}) \cdots K_k^*(p_{i_k}) K_{P(k)}(p_{i_k}) \cdots K_{P(1)}(p_{i_1}). \quad (6.40)
\end{aligned}$$

It is clear from Eq. (6.40) that if  $k > n$ , the correlation function  $G^{(k)}$  is equal to zero — as there are only  $n$  occupied modes, there cannot be  $k$  different occupied modes  $p_{i_1}, \dots, p_{i_k}$ . This must be so because if there are only  $n$  electrons in the system, the probability of finding more than  $n$  electrons is zero.

In the sum in Eq. (6.40) miss the combinations of indexes  $i_1, \dots, i_k$ , in which some indexes are equal. However, adding such combinations to the sum does not change the right-hand side of Eq. (6.40) because if some two indexes  $i_a, i_b$  are equal, the summation over the permutations  $P$  gives zero. Therefore we can write

$$G^{(k)} = \sum_{i_1, \dots, i_k=1}^n \sum_{P \in \mathcal{P}} \text{sign}(P) K_1^*(p_{i_1}) \cdots K_k^*(p_{i_k}) K_{P(k)}(p_{i_k}) \cdots K_{P(1)}(p_{i_1}). \quad (6.41)$$

The problem is now to evaluate the sum over the indexes  $i_1, \dots, i_k$ . For this purpose we define the function  $A(p)$  in the following way:

$$A(p) = \begin{cases} 1 & \text{if } p \in \{p_1, \dots, p_n\} \\ 0 & \text{if } p \notin \{p_1, \dots, p_n\} \end{cases}. \quad (6.42)$$

Alternatively,  $A(p)$  can be expressed as  $\sum_{i=1}^n \delta(p, p_i)$ .

It can be shown that for any function  $F$  it holds

$$\sum_{i_1, \dots, i_k=1}^n F(p_{i_1}, \dots, p_{i_k}) = \sum_{q_1, \dots, q_k} A(q_1) A(q_2) \cdots A(q_k) F(q_1, \dots, q_k). \quad (6.43)$$

To prove this, we evaluate the right-hand side of this equation using the alternative definition of  $A(p)$ :

$$\begin{aligned}
&\sum_{q_1, \dots, q_k} A(q_1) A(q_2) \cdots A(q_k) F(q_1, \dots, q_k) \\
&= \sum_{q_1, \dots, q_k} \sum_{i_1=1}^n \delta(q_1, p_{i_1}) \cdots \sum_{i_k=1}^n \delta(q_k, p_{i_k}) F(q_1, \dots, q_k) = \sum_{i_1, \dots, i_k=1}^n F(p_{i_1}, \dots, p_{i_k}). \quad (6.44)
\end{aligned}$$

Using the formula (6.43) in Eq. (6.41), we get

$$\begin{aligned}
G^{(k)} &= \sum_{q_1, \dots, q_k} A(q_1) \cdots A(q_k) \sum_{P \in \mathcal{P}} \text{sign}(P) K_1^*(q_1) \cdots K_k^*(q_k) K_{P(k)}(q_k) \cdots K_{P(1)}(q_1) \\
&= \sum_{P \in \mathcal{P}} \text{sign}(P) \sum_q A(q) K_1^*(q) K_{P(1)}(q) \cdots \sum_q A(q) K_k^*(q) K_{P(k)}(q). \quad (6.45)
\end{aligned}$$



If we denote  $\Gamma_{ij} = \sum_q A(q) K_i^*(q) K_j(q)$ , this result can be written precisely as for the chaotic state in the form of a determinant

$$G^{(k)} = \begin{vmatrix} \Gamma_{11} & \Gamma_{12} & \dots & \Gamma_{1k} \\ \Gamma_{21} & \Gamma_{22} & \dots & \Gamma_{2k} \\ \vdots & \vdots & & \vdots \\ \Gamma_{k1} & \Gamma_{k2} & \dots & \Gamma_{kk} \end{vmatrix}. \quad (6.46)$$

The  $\Gamma_{ij}$  expresses again the cross correlation function, i.e.,  $\Gamma_{ij} = \text{Tr}\{\hat{\rho}\hat{\psi}^+(\mathbf{r}_i, t_i)\hat{\psi}(\mathbf{r}_j, t_j)\}$ , which can be proved in a complete analogy with the derivation of  $G^{(k)}$ .

It is reasonable to introduce the one-electron momentum distribution for our system, especially if the number  $n$  of electrons is large. The probability that a randomly chosen electron has the momentum  $p$  is evidently equal to  $1/n$  if  $p \in \{p_1, \dots, p_n\}$  and zero otherwise. Thus the momentum distribution  $f(p)$  can be expressed easily with the help of  $A(p)$  as

$$f(p) = \frac{A(p)}{n}. \quad (6.47)$$

Note that  $\sum_p f(p) = 1$ , which must hold for a probability distribution. Substituting Eq. (6.47) into Eq. (6.45), we get

$$G^{(k)} = n^k \sum_{P \in \mathcal{P}} \text{sign}(P) \sum_q f(q) K_1^*(q) K_{P(1)}(q) \cdots \sum_q f(q) K_k^*(q) K_{P(k)}(q), \quad (6.48)$$

which is the  $k$ -electron correlation function expressed in terms of the one-particle momentum distribution.

Comparing Eq. (6.48) with Eq. (6.26), or Eq. (6.46) with Eq. (6.28), we see that the form of the correlation functions for even very different states can be the same. This might suggest that the nature of correlations in electron beams is quite universal and maybe not so much sensitive to the exact form of the state in question.

## Appendix A: The trace in Eq. (6.19)

As we have seen, the trace of the operator

$$|p_1, \dots, p_n\rangle \langle p_1, \dots, p_n| \hat{a}^+(q_1) \hat{a}^+(q_2) \hat{a}(q'_2) \hat{a}(q'_1) \quad (6.49)$$

enters the calculation of the  $k$ -electron correlation function in a very important way. The evaluation of the trace is based in principle on the anticommutation relations (4.13) holding for the electron creation and annihilation operators. Now we will perform this evaluation, but instead of using a very formal way issuing directly from the anticommutation relations, we will rather explain the calculation more intuitively. The method will be shown first on the case  $k = 2$  and then generalized on a case of an arbitrary  $k$ .

### The trace for $k = 2$

If  $\langle 2|$  and  $|1\rangle$  are some states in the Hilbert space, the following identity holds:

$$\text{Tr}(|1\rangle\langle 2|) = \langle 2|1\rangle. \quad (6.50)$$

This is generally used when one goes over from the state-vector description to the density-operator description of quantum states. Putting  $|1\rangle = |p_1, \dots, p_n\rangle$  and  $\langle 2| = \langle p_1, \dots, p_n| \hat{a}^+(q_1) \hat{a}^+(q_2) \hat{a}(q'_2) \hat{a}(q'_1)$ , we get

$$\begin{aligned} \text{Tr} \left\{ |p_1, \dots, p_n\rangle \langle p_1, \dots, p_n| \hat{a}^+(q_1) \hat{a}^+(q_2) \hat{a}(q'_2) \hat{a}(q'_1) \right\} \\ = \langle p_1, \dots, p_n| \hat{a}^+(q_1) \hat{a}^+(q_2) \hat{a}(q'_2) \hat{a}(q'_1) |p_1, \dots, p_n\rangle. \end{aligned} \quad (6.51)$$

The right-hand side of Eq. (6.51) can be viewed as a scalar product  $\langle u|v\rangle$  of two vectors  $|u\rangle = \hat{a}(q_2) \hat{a}(q_1) |p_1, \dots, p_n\rangle$  and  $|v\rangle = \hat{a}(q'_2) \hat{a}(q'_1) |p_1, \dots, p_n\rangle$ . Because of orthonormality of the state vectors corresponding to different sets of modes, this product vanishes unless precisely the same modes in both the vectors  $|u\rangle$  and  $|v\rangle$  are occupied. The vector  $|u\rangle$  has been obtained from  $|p_1, \dots, p_n\rangle$  by taking away two electrons in modes  $q_1, q_2$  and similarly, the vector  $|v\rangle$  has been obtained from  $|p_1, \dots, p_n\rangle$  by taking away two electrons in modes  $q'_1, q'_2$ . Therefore the product  $\langle u|v\rangle$  vanishes unless

$$\text{either } q_1 = q'_1, \quad q_2 = q'_2, \quad (6.52)$$

$$\text{or } q_1 = q'_2, \quad q_2 = q'_1. \quad (6.53)$$

In the first case (6.52) we have  $|u\rangle = |v\rangle$  and therefore  $\langle u|v\rangle = 1$ . In the second case (6.53), however, due to the anticommutation relation  $\hat{a}(q_1) \hat{a}(q_2) = -\hat{a}(q_2) \hat{a}(q_1)$  holding for the annihilation operators we obtain

$$|v\rangle = \hat{a}(q'_2) \hat{a}(q'_1) |p_1, \dots, p_n\rangle = \hat{a}(q_1) \hat{a}(q_2) |p_1, \dots, p_n\rangle = -\hat{a}(q_2) \hat{a}(q_1) |p_1, \dots, p_n\rangle = -|u\rangle \quad (6.54)$$

and therefore  $\langle u|v\rangle = -1$ . So if  $|u\rangle$  and  $|v\rangle$  are both different from the zero vector, it is not difficult to express the product  $\langle u|v\rangle$  with the help of Kronecker deltas:

$$\langle u|v\rangle = \delta(q_1, q'_1) \delta(q_2, q'_2) - \delta(q_1, q'_2) \delta(q_2, q'_1). \quad (6.55)$$

Now, it can happen that some of the modes  $q_1, q_2, q'_1, q'_2$  are not contained in the set  $\{p_1, \dots, p_n\}$ . Then, of course, one or both of  $|u\rangle, |v\rangle$  are equal to the zero vector and the scalar product  $\langle u|v\rangle$  is hence equal to zero. It is not necessary to investigate whether all the four modes  $q_1, q_2, q'_1, q'_2$  are contained among  $p_1, \dots, p_n$  because if one of the conditions (6.52), (6.53) is fulfilled, from  $q_1, q_2 \in \{p_1, \dots, p_n\}$  it follows that also  $q'_1, q'_2 \in \{p_1, \dots, p_n\}$ . It is again easy to write down an expression that is equal to unity if  $q_1, q_2 \in \{p_1, \dots, p_n\}$  and zero otherwise:

$$\sum_{\substack{i,j=1 \\ i \neq j}}^n \delta(q_1, p_i) \delta(q_2, p_j). \quad (6.56)$$

The terms with  $i = j$  are omitted in the sum because they would give nonzero results for  $q_1 = q_2$ , which cannot be because  $\hat{a}(q_1)\hat{a}(q_1) = 0$  for fermions.

Altogether, we see that it is possible to express the scalar product  $\langle u|v \rangle$  with the help of (6.55) and (6.56) as

$$\langle u|v \rangle = \left[ \delta(q_1, q'_1)\delta(q_2, q'_2) - \delta(q_1, q'_2)\delta(q_2, q'_1) \right] \sum_{\substack{i,j=1 \\ i \neq j}}^n \delta(q_1, p_i)\delta(q_2, p_j). \quad (6.57)$$

### The trace for arbitrary $k$

If we now compute the trace in Eq. (6.19) for an arbitrary  $k$ , we proceed in the completely analogous way: however, instead of the two conditions (6.52), (6.53) we have now  $k!$  conditions, corresponding to  $k!$  permutations of  $k$  indexes  $1, 2, \dots, k$ . The condition corresponding to the permutation  $P$  can be expressed as

$$q_1 = q'_{P(1)}, \quad q_2 = q'_{P(2)}, \quad \dots, \quad q_k = q'_{P(k)}. \quad (6.58)$$

The term expressing whether the sets  $\{q_1, \dots, q_k\}$  and  $\{q'_1, \dots, q'_k\}$  contain the same modes<sup>2</sup> looks therefore as

$$\sum_{P \in \mathcal{P}} \text{sign}(P) \delta(q_1, q'_{P(1)}) \cdots \delta(q_k, q'_{P(k)}) \quad (6.59)$$

On the other hand, the expression that is equal to unity if all the modes  $q_1, \dots, q_k$  are contained among  $p_1, \dots, p_n$ , and equal to zero otherwise, can be in analogy with Eq. (6.56) written as

$$\sum_{\substack{i_1, \dots, i_k=1 \\ \text{all } i \text{ different}}}^n \delta(p_{i_1}, q_1) \cdots \delta(p_{i_k}, q_k). \quad (6.60)$$

Combining the expressions (6.59) and (6.60), we finally arrive at

$$\begin{aligned} & \langle p_1, \dots, p_n | \hat{a}^+(q_1) \cdots \hat{a}^+(q_k) \hat{a}(q'_1) \cdots \hat{a}(q'_k) | p_1, \dots, p_n \rangle \\ &= \sum_{P \in \mathcal{P}} \text{sign}(P) \delta(q_1, q'_{P(1)}) \cdots \delta(q_k, q'_{P(k)}) \sum_{\substack{i_1, \dots, i_k=1 \\ \text{all } i \text{ different}}}^n \delta(p_{i_1}, q_1) \cdots \delta(p_{i_k}, q_k), \end{aligned} \quad (6.61)$$

which we wanted to prove.

## Appendix B: Derivation of the cross-correlation function for the chaotic state

With the help of the relations (4.11) and (4.12) we express the operator  $\hat{\psi}_i^\dagger \hat{\psi}_j = \hat{\psi}^\dagger(\mathbf{r}_i, t_i) \hat{\psi}(\mathbf{r}_j, t_j)$  as follows:

$$\hat{\psi}_i^\dagger \hat{\psi}_j = \sum_{q, q'} K_i^*(q) K_j(q') \hat{a}^\dagger(q) \hat{a}(q'). \quad (6.62)$$

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<sup>2</sup>the term is equal to unity if the two sets are the same and equal to zero otherwise

Then the cross correlation function is equal to

$$\begin{aligned} \text{Tr}(\rho_n \hat{\psi}_i^\dagger \hat{\psi}_j) &= \sum_{q, q'} \sum'_{p_1, \dots, p_n} f(p_1) \cdots f(p_n) K_i^*(q) K_j(q') \text{Tr}\{|p_1, \dots, p_n\rangle \langle p_1, \dots, p_n| \hat{a}^\dagger(q) \hat{a}(q')\} \\ &= \sum_{q, q'} \sum'_{p_1, \dots, p_n} f(p_1) \cdots f(p_n) K_i^*(q) K_j(q') \delta(q, q') \sum_{s=1}^n \delta(q, p_s). \end{aligned} \quad (6.63)$$

After the evaluation of the sums over  $p_1, \dots, p_n$  and  $q'$  in a way analogous to the one used during the calculation of the correlation function, we obtain

$$\text{Tr}(\rho_n \hat{\psi}_i^\dagger \hat{\psi}_j) = n \sum_q f(q) K_i^*(q) K_j(q), \quad (6.64)$$

and with the help of Eq. (6.5) we get the desired result

$$\text{Tr}(\rho \hat{\psi}_i^\dagger \hat{\psi}_j) = N \sum_q f(q) K_i^*(q) K_j(q) = \Gamma_{ij}. \quad (6.65)$$

# Chapter 7

## Correlation functions for partially polarized beams

Up to now we have considered the case of spin-polarized electrons. This is of course a situation we hardly meet in experiment where we deal mostly with just partially polarized or totally unpolarized beams. To make a precise description of such a situation and calculate the correlation functions, it would be necessary to index the field operators (4.7) not only with the variables  $\mathbf{r}$  and  $t$  but also with the spin variable. The correlation functions could be then defined either with the consideration of the spin or without it. This way, we could calculate e.g. the probability of detecting  $k$  electrons at the space-time points  $(\mathbf{r}_i, t_i)$  ( $i = 1, \dots, k$ ) with particular spin orientations at each point. However, in a real experiment we are glad to detect an electron and have not much thinking of measuring its spin at the same time. Therefore we are interested in the detection probability regardless to the spin orientation, which is given by the “spinless” correlation function. In this chapter we will show a heuristic derivation of such a correlation function for a partially polarized electron beam using simple argumentation and avoiding the above mentioned definitions of the spin field operators. We will show that it is possible to express the  $k$ -electron correlation function with the help of correlation functions for a polarized beam of all orders up to  $k$ .

### 7.1 Independence of two spin components

We say that a beam is polarized with respect to an axis  $\mathbf{a}$  if the spins of all electrons in the beam have the directions of this axis. If  $|1\rangle$  and  $|2\rangle$  denote the one-electron spin states “up” and “down” with respect to the axis  $\mathbf{a}$ , the one-electron spin density operator of such a beam is simply

$$\hat{\rho}_s = |1\rangle\langle 1|. \quad (7.1)$$

If the beam is only partially polarized, the corresponding spin density operator describes no more a pure state and its diagonal form is rather

$$\hat{\rho}_s = \rho_1|1\rangle\langle 1| + \rho_2|2\rangle\langle 2| = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}, \quad (7.2)$$

where  $\rho_1$  and  $\rho_2$  ( $\rho_1 + \rho_2 = 1$ ) are the probabilities that the spin is oriented up and down, respectively. The fact that the density operator is diagonal means that there is no correlation between the two spin components up and down. This is very important: as there is no correlation between the two components, the emission of partially polarized electrons by the source is equivalent to an emission from two independent spin-polarized electron sources. Both of these two “new” sources are of course located at the place of the original source and their emission intensities are equal to  $\rho_1$  and  $\rho_2$  times the intensity of the original source, respectively.

## 7.2 Two-electron correlation function

The two-electron correlation function expresses the probability of the event of finding two electrons at the points  $(\mathbf{r}_1, t_1)$  and  $(\mathbf{r}_2, t_2)$ . This event can be divided into four cases that are distinguishable in principle because the coordinate and spin operators mutually commute:

1. the spins of both electrons are oriented up
2. the spins of both electrons are oriented down
3. the spin of the electron at  $(\mathbf{r}_1, t_1)$  is oriented down, the spin of the electron at  $(\mathbf{r}_2, t_2)$  is oriented up
4. the spin of the electron at  $(\mathbf{r}_1, t_1)$  is oriented up, the spin of the electron at  $(\mathbf{r}_2, t_2)$  is oriented down.

The two-electron correlation function in the cases 1. and 2. is simply equal to the analogous correlation function  $G_{12}^{(2)}$  for polarized electrons because we have two electrons from the same polarized source. On the other hand, in the cases 3. and 4. we deal with two electrons from two independent, oppositely polarized sources. Therefore the electrons are uncorrelated and the correlation function is equal to the product of one-electron correlation functions, i.e.,  $G_1^{(1)}G_2^{(1)}$ . As the probability that one electron is polarized up or down is  $\rho_1$  or  $\rho_2$ , respectively, the probabilities of the cases 1. — 4. are evidently  $\rho_1^2$ ,  $\rho_2^2$ ,  $\rho_1\rho_2$ , and  $\rho_1\rho_2$ , respectively. The total correlation function can be then written as the weighed average

$$O_{12}^{(2)} = (\rho_1^2 + \rho_2^2)G_{12}^{(2)} + 2\rho_1\rho_2G_1^{(1)}G_2^{(1)} \quad (7.3)$$

(we denote it as  $O_{12}^{(2)}$  to distinguish it from the correlation function  $G_{12}^{(2)}$  for polarized electrons).

We can now express this result for the chaotic state. So, after substituting  $G_{12}^{(2)} = G_1^{(1)}G_2^{(1)}(1 - |\gamma_{12}|^2)$  according to Eq. (6.30) into Eq. (7.3) we obtain

$$O_{12}^{(2)} = (\rho_1^2 + \rho_2^2) G_1^{(1)}G_2^{(1)}(1 - |\gamma_{12}|^2) + 2\rho_1\rho_2 G_1^{(1)}G_2^{(1)} = G_1^{(1)}G_2^{(1)} [1 - (\rho_1^2 + \rho_2^2)] |\gamma_{12}|^2. \quad (7.4)$$

It is very convenient to introduce the degree of polarization  $P$ <sup>1</sup>,

$$P = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} = \rho_1 - \rho_2 \quad (7.5)$$

(we suppose that  $\rho_1 \geq \rho_2$ ), which is equal to unity for a completely polarized beam ( $\rho_1 = 1, \rho_2 = 0$ ) and equal to zero for a completely unpolarized beam ( $\rho_1 = \rho_2 = 1/2$ ). Using Eq. (7.5) and the normalization condition  $\rho_1 + \rho_2 = 1$ , we can express  $\rho_1, \rho_2$  in terms of the degree of polarization  $P$ :

$$\rho_1 = \frac{1+P}{2}, \quad \rho_2 = \frac{1-P}{2}. \quad (7.6)$$

Substituting this into Eq. (7.4), we obtain

$$O_{12}^{(2)} = G_1^{(1)}G_2^{(1)} \left( 1 - \frac{1+P^2}{2} |\gamma_{12}|^2 \right). \quad (7.7)$$

This equation shows that the correlation, given by the second term in the parentheses, is still present for partially polarized electrons but its “strength” is only  $(1+P^2)/2$  times the strength of the correlation for polarized electrons. For example, if we have a 50% polarized beam, the correlation is reduced by the factor 0.625, and for a completely unpolarized beam for which  $P = 0$ , the correlation reduces to one half.

It is worth noting that in quantum optics the same result has been obtained – for thermal light with the degree of polarization  $P$ , the correlation (in this case bunching) is also reduced by the factor  $(1+P^2)/2$  with respect to polarized light (see [1], page 432).

### 7.3 The $k$ -electron correlation function

Now, if we deal with  $k$  electrons at  $k$  points  $(\mathbf{r}_1, t_1), \dots, (\mathbf{r}_k, t_k)$ , there are  $2^k$  possibilities how they can be polarized. We denote each of them by the sequence  $s_1, s_2, \dots, s_k$ , every  $s_i$  expressing the spin polarization of the electron at the point  $(\mathbf{r}_i, t_i)$  and having one of two possible values, 1 for spin up and or 2 for spin down. The probability  $P(s_1, \dots, s_k)$  that the electrons have polarizations  $s_1, \dots, s_k$  is evidently equal to  $\rho_1^{k_1} \rho_2^{k_2}$ , where  $k_1$  and

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<sup>1</sup>The degree of polarization is defined using the parameters  $\rho_1, \rho_2$  for the axis for which the difference  $\rho_1 - \rho_2$  is maximal. The proof that precisely this is the axis for which the spin density operator is diagonal is given in Appendix, page 41.

$k_2$  expresses how many times there appears 1 and 2 among the numbers  $s_1, \dots, s_k$ , respectively. If  $O_{1,\dots,k}^{(k)}(s_1, \dots, s_k)$  denotes the  $k$ -electron correlation function for this particular spin combination, the total  $k$ -electron correlation function can be written as

$$O_{1,\dots,k}^{(k)} = \sum_{s_1, \dots, s_k} P(s_1, \dots, s_k) O_{1,\dots,k}^{(k)}(s_1, \dots, s_k), \quad (7.8)$$

the sum being made over all the possibilities  $s_1, \dots, s_k$ . Now, if the spin polarizations of the electrons are  $s_1, \dots, s_k$ , the situation is the same as if we dealt with two independent sets of electrons — one set of  $k_1$  up-polarized electrons originating from the first source and another set of  $k_2$  down-polarized electrons originating from the second source. The correlation function  $O_{1,\dots,k}^{(k)}(s_1, \dots, s_k)$  factorizes therefore into a product of two correlation functions for polarized electrons:

$$O_{1,\dots,k}^{(k)}(s_1, \dots, s_k) = G^{(k_1)}(\{r, t\}_{\text{up}}) G^{(k_2)}(\{r, t\}_{\text{down}}). \quad (7.9)$$

Here  $\{r, t\}_{\text{up}}$  and  $\{r, t\}_{\text{down}}$  denote the sets of points at which the electrons are polarized up and down, respectively. Using Eqs. (7.8) and (7.9), we can write a few first terms in the sum:

$$\begin{aligned} O_{1,\dots,k}^{(k)} &= (\rho_1^k + \rho_2^k) G_{1,\dots,k}^{(k)} \\ &\quad + (\rho_1^{k-1} \rho_2 + \rho_2^{k-1} \rho_1) (G_{2,\dots,k}^{(k-1)} G_1^{(1)} + G_{1,3,\dots,k}^{(k-1)} G_2^{(1)} + \dots + G_{1,\dots,k-1}^{(k-1)} G_k^{(1)}) \\ &\quad + (\rho_1^{k-2} \rho_2^2 + \rho_2^{k-2} \rho_1^2) (G_{3,\dots,k}^{(k-2)} G_{1,2}^{(2)} + G_{2,4,\dots,k}^{(k-2)} G_{1,3}^{(2)} \dots + G_{1,\dots,k-2}^{(k-2)} G_{k-1,k}^{(2)}) + \dots \end{aligned} \quad (7.10)$$

To see the structure of such a series better, let us write down the three- and four-electron correlation functions for illustration:

$$O_{1,2,3}^{(3)} = (\rho_1^3 + \rho_2^3) G_{1,2,3}^{(3)} + (\rho_1^2 \rho_2 + \rho_2^2 \rho_1) (G_{1,2}^{(2)} G_3^{(1)} + G_{1,3}^{(2)} G_2^{(1)} + G_{2,3}^{(2)} G_1^{(1)}), \quad (7.11)$$

$$\begin{aligned} O_{1,2,3,4}^{(4)} &= (\rho_1^4 + \rho_2^4) G_{1,2,3,4}^{(4)} \\ &\quad + (\rho_1^3 \rho_2 + \rho_2^3 \rho_1) (G_{1,2,3}^{(3)} G_4^{(1)} + G_{1,2,4}^{(3)} G_3^{(1)} + G_{1,3,4}^{(3)} G_2^{(1)} + G_{2,3,4}^{(3)} G_1^{(1)}) \\ &\quad + 2\rho_1^2 \rho_2^2 (G_{1,2}^{(2)} G_{3,4}^{(2)} + G_{1,3}^{(2)} G_{2,4}^{(2)} + G_{1,4}^{(2)} G_{2,3}^{(2)}). \end{aligned} \quad (7.12)$$

In this way the  $k$ -electron correlation function for partially polarized electrons is expressed in terms of the one-, two- etc. up to the  $k$ -electron correlation functions for polarized electrons and the degree of polarization (that is connected with  $\rho_1, \rho_2$  via the relations (7.6)). It is evident that for big  $k$  the function  $O_{1,\dots,k}^{(k)}$  does not reflect the  $k$ -particle correlation function  $G_{1,\dots,k}^{(k)}$  very much as soon as  $P$  differs even slightly from unity. For example, if we have  $k = 10$  and  $P = 0.9$ , the term  $(\rho_1^k + \rho_2^k)$  is equal to only about 0.6, so the function  $O_{1,\dots,k}^{(k)}$  contains only 60 percent information about the polarized ten-electron correlation function. The remaining 40 percent correspond to the correlation functions



of lower orders. Thus we see that if we do not have a beam with a very high degree of polarization, it is of very little use to try to measure the correlation functions of higher orders. However, we have been much too optimistic even in having the idea of measuring the ten-electron correlation function; in comparison to the extreme difficulty of such an experiment, making a 99,9%-polarized electron beam seems to be just a funny coffee-break problem.

## Appendix

As has been already mentioned, if one measures the projection of an electron spin to any axis, there are two possible results of this measurement, say “up” and “down”. If we denote  $q_1$  and  $q_2$  the probabilities of these results, respectively, there exists some axis for which the difference  $q_1 - q_2$  is maximal. The degree of polarization is then defined as  $P = (q_1 - q_2)/(q_1 + q_2) = q_1 - q_2$  for this particular axis. We will show now that this axis is identical with the axis  $\mathbf{a}$  with respect to which the one-electron spin density operator diagonalizes.

Let  $\{|1\rangle, |2\rangle\}$  denote an orthonormal basis in which the spin density matrix has a diagonal form

$$\hat{\rho} = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} = \rho_1|1\rangle\langle 1| + \rho_2|2\rangle\langle 2| \quad (7.13)$$

with  $\rho_1 > \rho_2$ . Any other orthonormal basis  $\{|a\rangle, |b\rangle\}$  in the two-dimensional Pauli spinor space can be expressed through the original basis in the general form

$$|a\rangle = u|1\rangle - v|2\rangle, \quad |b\rangle = e^{i\varphi}(v^*|1\rangle + u^*|2\rangle), \quad (7.14)$$

where  $u, v$  are complex numbers satisfying the condition  $|u|^2 + |v|^2 = 1$  and  $\varphi$  is a real number. The density matrix in the basis  $\{|a\rangle, |b\rangle\}$  has then the form

$$\hat{\rho} = \begin{pmatrix} |u|^2\rho_1 + |v|^2\rho_2 & e^{i\varphi}u^*v^*(\rho_1 - \rho_2) \\ e^{-i\varphi}uv(\rho_1 - \rho_2) & |u|^2\rho_2 + |v|^2\rho_1 \end{pmatrix}. \quad (7.15)$$

The absolute value of the difference of its diagonal elements is hence

$$U = \left| |u|^2\rho_1 + |v|^2\rho_2 - |u|^2\rho_2 - |v|^2\rho_1 \right| = \left| (|u|^2 - |v|^2) \right| (\rho_1 - \rho_2). \quad (7.16)$$

With the help of the condition  $|v|^2 = 1 - |u|^2$  this can be written as

$$U = \left| (2|u|^2 - 1) \right| (\rho_1 - \rho_2). \quad (7.17)$$

As  $0 \leq |u| \leq 1$  holds,  $U$  is maximal for  $|u| = 0 \Rightarrow |v| = 1$  or  $|u| = 1 \Rightarrow |v| = 0$ . In both of these cases the basis  $|a\rangle, |b\rangle$  coincides with the basis  $|1\rangle, |2\rangle$ , however (up to some phase factors or the coordinate permutation). Thus we see that the difference  $U$  is maximal for the basis in which  $\hat{\rho}$  has a diagonal form (7.13), that is, for the basis corresponding to the axis  $\mathbf{a}$  defined in section 7.1, and it therefore holds

$$P = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} = \rho_1 - \rho_2. \quad (7.18)$$

# Chapter 8

## Algebraic structure of correlation functions

In this chapter we will concentrate on interesting properties of the correlation functions of the chaotic state. Most of them exist because of the determinant form of the  $k$ -electron correlation function enables to apply the formalism of linear algebra to obtain very interesting general results. The argumentation in this chapter will be therefore a little bit more mathematical than usually but we hope the reader will not be bored about it.

As the key part of this chapter, we will prove a very important inequality that is fundamental for understanding the properties of the free-electron correlation functions. This inequality states that

$$G^{(l+m)}(\mathbf{r}_1, t_1, \dots, \mathbf{r}_{l+m}, t_{l+m}) \leq G^{(l)}(\mathbf{r}_1, t_1, \dots, \mathbf{r}_l, t_l) G^{(m)}(\mathbf{r}_{l+1}, t_{l+1}, \dots, \mathbf{r}_{l+m}, t_{l+m}), \quad (8.1)$$

or, in the shortened notation,

$$G_{1,2,\dots,l+m}^{(l+m)} \leq G_{1,2,\dots,l}^{(l)} G_{l+1,l+2,\dots,l+m}^{(m)}. \quad (8.2)$$

The proof is not trivial and we will need two supporting lemmas for it.

### **Lemma 1: Upper bound of the determinant of a positive-definite Hermitian matrix**

The determinant of any positive-definite Hermitian matrix  $A = (A_{ij})$  cannot exceed the product of the diagonal elements of  $A$ , i.e.,  $\det(A) \leq A_{11}A_{22} \cdots A_{kk}$ , and the equality takes place if and only if  $A$  is diagonal.

**Proof:** Thanks to the positive definiteness of  $A$ , all its diagonal elements  $A_{ii}$  are positive. Then we can define the matrix  $a = (a_{ij})$  with elements  $a_{ij} = A_{ij}/\sqrt{A_{ii}A_{jj}}$ . The matrix  $a$  is also Hermitian and its diagonal elements are equal to unity, so it has

the form

$$a = \begin{pmatrix} 1 & a_{12} & \dots & a_{1k} \\ a_{21} & 1 & \dots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \dots & 1 \end{pmatrix}. \quad (8.3)$$

Thanks to the hermiticity of the matrix  $a$ , it is possible to transform it into the diagonal form with a unitary transformation, i.e., there exists a unitary matrix  $U$  for which the matrix  $b = UaU^\dagger$  is diagonal. This transformation changes neither the determinant nor the trace of the matrix because it is a unitary transformation. If we denote the diagonal elements of the matrix  $b$  as  $b_i$ , then  $\text{Tr}(a) = \text{Tr}(b) = \sum_{i=1}^k b_i$  and  $\det(a) = \det(b) = \prod_{i=1}^k b_i$  evidently hold. At the same time, from Eq. (8.3) it follows that  $\text{Tr}(a) = k$ . To find out what is the maximal possible value of  $\det(a)$ , we will use now the inequality between the arithmetical and geometrical averages. The arithmetical average of the numbers  $b_i$  is  $\alpha = \sum_{i=1}^k b_i/k = 1$  and their geometrical average is  $\beta = \sqrt[k]{\prod_{i=1}^k b_i}$ . As the numbers  $b_i$  are positive, the inequality  $\beta \leq \alpha$  holds, from which it then follows that

$$\det(a) = \prod_{i=1}^k b_i \leq 1. \quad (8.4)$$

As is known, the equality  $\beta = \alpha$  takes place if and only if  $b_1 = b_2 = \dots = b_k$ . In this case the matrix  $b$  is the unit matrix, from which it follows that  $a$  is also the unit matrix and  $a_{ij} = \delta(i, j)$ . Thus,  $\det(a) \leq 1$  holds and the equality takes place only when all the non-diagonal elements of the matrix  $a$  vanish. Using the definition of  $a_{ij}$  and properties of determinants, we can express the determinant of the original matrix  $A$  with the help of  $\det(a)$  in the following way:

$$\det(A) = A_{11}A_{22} \cdots A_{kk} \det(a). \quad (8.5)$$

Then, from the inequality  $\det(a) \leq 1$  it follows

$$\det(A) \leq A_{11}A_{22} \cdots A_{kk}. \quad (8.6)$$

Moreover,  $A$  is diagonal if and only if  $a$  is diagonal. Therefore the equality in (8.6) takes place if and only if the matrix  $A$  is diagonal.

**Lemma 2: Hermiticity and definiteness of the matrix  $\Gamma^{(k)}$**

The matrix of the cross-correlation functions

$$\Gamma^{(k)} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \dots & \Gamma_{1k} \\ \Gamma_{21} & \Gamma_{22} & \dots & \Gamma_{2k} \\ \vdots & \vdots & & \vdots \\ \Gamma_{k1} & \Gamma_{k2} & \dots & \Gamma_{kk} \end{pmatrix} \quad (8.7)$$

is Hermitian and either positive-definite or positive-semidefinite.

**Proof:** The hermiticity of  $\Gamma^{(k)}$  follows simply from the fact that  $\Gamma_{ij} = \langle \hat{\psi}_i^\dagger \hat{\psi}_j \rangle = \langle \hat{\psi}_j^\dagger \hat{\psi}_i \rangle^* = \Gamma_{ji}^*$ . The proof of the second property will be based on the proof of an analogous statement for the case of photons in [1], p. 585. Let  $\hat{O}$  be the operator defined as

$$\hat{O} = \sum_{i=1}^k \lambda_i \hat{\psi}_i, \quad (8.8)$$

where  $\lambda_1, \dots, \lambda_k$  are arbitrary complex numbers. It is trivial to show that the quantum-mechanical average of the operator  $\hat{O}^\dagger \hat{O}$  is a non-negative number. At the same time, it holds

$$\langle \hat{O}^\dagger \hat{O} \rangle = \sum_{i,j=1}^k \lambda_i^* \lambda_j \langle \hat{\psi}_i^\dagger \hat{\psi}_j \rangle = \sum_{i,j=1}^k \lambda_i^* \lambda_j \Gamma_{ij}. \quad (8.9)$$

As the right-hand side of this equation is at the same time a quadratic form in the  $\lambda$ 's, the matrix  $\Gamma^{(k)}$  formed by the numbers  $\Gamma_{ij}$  must be either positive-definite or positive-semidefinite.

## 8.1 Proof of inequality (8.2)

Now, we can go back to the proof of the inequality (8.2). First, we will define two matrices of the type  $(l+m)/(l+m)$ : the matrix  $\Gamma$  composed simply of the cross-correlation functions  $\Gamma_{ij}$

$$\Gamma = \begin{pmatrix} \Gamma_{1,1} & \dots & \Gamma_{1,l+m} \\ \vdots & & \vdots \\ \Gamma_{l+m,1} & \dots & \Gamma_{l+m,l+m} \end{pmatrix} \quad (8.10)$$

and the matrix  $\Gamma'$  in the block form

$$\Gamma' = \begin{pmatrix} \Gamma^{(l)} & 0 \\ 0 & \Gamma^{(m)} \end{pmatrix}. \quad (8.11)$$

Here 0 stands for the zero matrices of the type  $l/m$  or  $m/l$  and  $\Gamma^{(l)}, \Gamma^{(m)}$  are the matrices of the type  $l/l$  and  $m/m$ , respectively, corresponding to the correlation functions  $G_{1,\dots,l}^{(l)}$  and  $G_{l+1,\dots,l+m}^{(m)}$ :

$$\Gamma^{(l)} = \begin{pmatrix} \Gamma_{1,1} & \dots & \Gamma_{1,l} \\ \vdots & & \vdots \\ \Gamma_{l,1} & \dots & \Gamma_{l,l} \end{pmatrix}, \quad \Gamma^{(m)} = \begin{pmatrix} \Gamma_{l+1,l+1} & \dots & \Gamma_{l+1,l+m} \\ \vdots & & \vdots \\ \Gamma_{l+m,l+1} & \dots & \Gamma_{l+m,l+m} \end{pmatrix} \quad (8.12)$$

So, the matrices  $\Gamma$  and  $\Gamma'$  differ only in the two non-diagonal right-angle parts of the type  $l/m$  and  $m/l$ . Due to Eq. (6.28) and the block form of  $\Gamma$  it holds

$$G_{1,\dots,l+m}^{(l+m)} = \det(\Gamma), \quad G_{1,\dots,l}^{(l)} G_{l+1,\dots,l+m}^{(m)} = \det(\Gamma^{(l)}) \det(\Gamma^{(m)}) = \det(\Gamma'). \quad (8.13)$$

Now, we know that the matrix  $\Gamma$  is either positive-definite or positive-semidefinite. In the latter case, the inequality (8.2) is satisfied trivially because then  $\det(\Gamma) = 0$  and  $\det(\Gamma^{(l)})$ ,  $\det(\Gamma^{(m)})$  are both non-negative due to Lemma 2. Therefore in the following we will discuss the case when  $\Gamma$  is positive-definite.

As the matrices  $\Gamma^{(l)}$  and  $\Gamma^{(m)}$  are Hermitian, it is possible to transform each of them into the diagonal form with a unitary transformation. Let  $U^{(l)}$  and  $U^{(m)}$  denote the corresponding unitary transformational matrices, so that the matrices  $D^{(l)} = U^{(l)}\Gamma^{(l)}U^{(l)\dagger}$  and  $D^{(m)} = U^{(m)}\Gamma^{(m)}U^{(m)\dagger}$  are both diagonal. Then evidently the unitary matrix

$$U = \begin{pmatrix} U^{(l)} & 0 \\ 0 & U^{(m)} \end{pmatrix} \quad (8.14)$$

transforms the matrix  $\Gamma'$  into the diagonal form, so that  $D' = U\Gamma'U^\dagger$  is diagonal. Let  $D$  denote the matrix obtained from  $\Gamma$  by the same unitary transformation, i.e., let  $D = U\Gamma U^\dagger$ . Thanks to the block form of the matrix  $U$ , the matrix  $D$  has the form

$$D = \begin{pmatrix} D^{(l)} & D^{(lm)} \\ D^{(ml)} & D^{(m)} \end{pmatrix}, \quad (8.15)$$

where  $D^{(lm)}$  and  $D^{(ml)}$  are some mutually Hermite-conjugate matrices of the type  $l/m$  and  $m/l$ , respectively. Applying now Lemma 1 to the matrix  $D$  (we can do that because  $D$  is positive-definite and Hermitian; the latter follows from the unitarity of the matrix  $U$ ), we see that  $\det(D) \leq \det(D')$  because the diagonal elements of the matrices  $D$  and  $D'$  are identical and  $D'$  is diagonal. Combining this with the equations that hold due to the unitarity of the matrix  $U$ ,

$$\det(D) = \det(\Gamma), \quad \det(D') = \det(\Gamma') = \det(\Gamma^{(l)})\det(\Gamma^{(m)}), \quad (8.16)$$

and with Eq. (8.13), we finally obtain the inequality (8.2). Now, inequality  $\det(D) \leq \det(D')$  changes into equality if and only if the matrix  $D$  is diagonal, i.e., if  $D^{(lm)}$  and  $D^{(ml)}$  are the zero matrices. Then, again due to the block form of the transformation matrix  $U$ , also the matrices

$$\Gamma^{(lm)} = \begin{pmatrix} \Gamma_{1,l+1} & \cdots & \Gamma_{1,l+m} \\ \vdots & & \vdots \\ \Gamma_{l,l+1} & \cdots & \Gamma_{l,l+m} \end{pmatrix}, \quad \Gamma^{(ml)} = \begin{pmatrix} \Gamma_{l+1,1} & \cdots & \Gamma_{l+1,l} \\ \vdots & & \vdots \\ \Gamma_{l+m,1} & \cdots & \Gamma_{l+m,l} \end{pmatrix} \quad (8.17)$$

are the zero matrices. Thus we can conclude that the inequality (8.2) holds and changes into equality if and only if all the complex degrees of coherence  $\Gamma_{i,j}$  vanish for  $i = 1, \dots, l$  and  $j = l+1, \dots, l+m$ .

## 8.2 Fermionic nature of electron correlations

The inequality (8.2) that we have just proved says that the probability of finding  $l+m$  electrons at the points  $(\mathbf{r}_1, t_1), \dots, (\mathbf{r}_{l+m}, t_{l+m})$  is always less or equal to the probability of finding  $l$  electrons at the points  $(\mathbf{r}_1, t_1), \dots, (\mathbf{r}_l, t_l)$  multiplied by the probability

of finding the remaining  $m$  electrons at the points  $(\mathbf{r}_{l+1}, t_{l+1}), \dots, (\mathbf{r}_{l+m}, t_{l+m})$ . This is a fundamental statement that expresses the fermion behaviour of electrons in an especially compact way. The inequality (8.2) can be of course applied repeatedly and the points  $(\mathbf{r}_1, t_1), \dots, (\mathbf{r}_{l+m}, t_{l+m})$  can be arbitrarily interchanged to obtain even more inequalities. We will write a few examples for illustration:

$$G_{1,2}^{(2)} \leq G_1^{(1)} G_2^{(1)} \quad (8.18)$$

$$G_{1,2,3}^{(3)} \leq G_1^{(1)} G_{2,3}^{(2)} \quad (8.19)$$

$$G_{1,2,3,4}^{(4)} \leq G_{1,4}^{(2)} G_{2,3}^{(2)} \quad (8.20)$$

$$G_{1,2,3,4,5,6,7}^{(7)} \leq G_{1,2}^{(2)} G_{3,5,7}^{(3)} G_{4,6}^{(2)} \quad (8.21)$$

$$G_{1,2,\dots,k}^{(k)} \leq G_1^{(1)} G_2^{(1)} \dots G_k^{(1)}. \quad (8.22)$$

For example, Eq. (8.22) expresses the fact that the probability that at each of the  $k$  space-time points we find an electron cannot exceed the product of the probabilities of finding an electron at the individual points.

As we have seen, the case of equality in Eq. (8.2) corresponds to the situation when  $\Gamma_{ij} = 0$  for all  $i = 1, \dots, l$  and  $j = l + 1, \dots, l + m$ . Let find out what these conditions really mean. First, we recall the definition of the complex degree of coherence given in Chap. (6),

$$\gamma_{ij} = \frac{\Gamma_{ij}}{\sqrt{\Gamma_{ii}\Gamma_{jj}}}. \quad (8.23)$$

An analogous physical quantity has been known in optics for a long time that expresses the mutual coherence of the electromagnetic field at two space-time points. Similarly,  $\gamma_{ij}$  expresses the mutual coherence of the electron field at the space-time points  $(\mathbf{r}_i, t_i)$  and  $(\mathbf{r}_j, t_j)$  and contains information about both the temporal and spatial coherence of the field. From the non-negative definiteness of the matrix  $\Gamma$  it follows that  $\Gamma_{ij}\Gamma_{ji} \leq \Gamma_{ii}\Gamma_{jj}$ , which yields that  $|\gamma_{ij}| \leq 1$  for all  $i, j$ . The case  $|\gamma_{ij}| = 1$  corresponds to the complete mutual coherence of the electron field at the points  $(\mathbf{r}_i, t_i), (\mathbf{r}_j, t_j)$ , while  $|\gamma_{ij}| = 0$  corresponds to the complete incoherence. So for  $|\gamma_{ij}| = 0$ , even if we know the properties of the field at the point  $\mathbf{r}_i$  at the time  $t_i$  completely, we cannot say anything about the field at the point  $\mathbf{r}_j$  at the time  $t_j$ . On the other hand, if  $|\gamma_{ij}| > 0$ , some properties of the field at the point  $\mathbf{r}_j$  at the time  $t_j$  can be determined from the knowledge of the electron field at the point  $\mathbf{r}_i$  at the time  $t_i$ .

Thus, the conditions  $\Gamma_{ij} = 0$  for all  $i = 1, \dots, l$  and  $j = l + 1, \dots, l + m$  express the fact that the electron field at any point of the first set  $S_l = \{(\mathbf{r}_i, t_i) | i = 1, \dots, l\}$  is incoherent

with the field at any point of the second set  $S_m = \{(\mathbf{r}_j, t_j) | j = l+1, \dots, l+m\}$ . The equality in (8.2) is then very reasonable: if the fields at the points corresponding to the both sets  $S_l, S_m$  are mutually completely incoherent, the probability of finding an electron at each of the  $l + m$  points factorizes into the product of the probabilities corresponding to the individual sets. On the other hand, if there exists at least one pair of points  $(\mathbf{r}_i, t_i) \in S_l, (\mathbf{r}_j, t_j) \in S_m$  for which  $\gamma_{ij}$  is nonzero, then the inequality (8.2) is sharp. We can say that the electrons “do not like to stay together” in some sense. Indeed, if  $|\gamma_{ij}| > 0$  holds, then the electron field at the point  $(\mathbf{r}_i, t_i)$  “knows” about the field at the point  $(\mathbf{r}_j, t_j)$  and if there is an electron at  $(\mathbf{r}_i, t_i)$ , another electron does not “like” very much to be at  $(\mathbf{r}_j, t_j)$ . This “repulsion”, or better said, “exchange repulsion”, leads then to the sharp inequality in (8.2). In the extreme case when  $|\gamma_{ij}| = 1$ , the subdeterminant  $\begin{vmatrix} \Gamma_{ii} & \Gamma_{ij} \\ \Gamma_{ji} & \Gamma_{jj} \end{vmatrix}$  is equal to zero and therefore the matrix  $\Gamma$  is positive-semidefinite. The correlation function  $G_{1, \dots, l+m}^{(l+m)}$ , which is the determinant of  $\Gamma$ , turns then into zero. This means that if  $|\gamma_{ij}| = 1$ , no two electrons can be found at the space-time points  $(\mathbf{r}_i, t_i)$  and  $(\mathbf{r}_j, t_j)$  simultaneously and therefore of course no  $l + m$  electrons can be found at the points  $(\mathbf{r}_1, t_1), \dots, (\mathbf{r}_{l+m}, t_{l+m})$  that include also the points  $(\mathbf{r}_i, t_i)$  and  $(\mathbf{r}_j, t_j)$ .

Let us compare this result with the Pauli principle: the Pauli principle prohibits two fermions to be in the same quantum state, while our result prohibits two electrons to be at space-time points  $(\mathbf{r}_i, t_i)$  and  $(\mathbf{r}_j, t_j)$  for which  $|\gamma_{ij}| = 1$ . The similarity of both the statements is obvious.

### 8.3 Correlation function corresponding to new creation and annihilation operators

Consider now a general unitary transformation that transforms the creation and annihilation operators  $\hat{\psi}_1^\dagger, \dots, \hat{\psi}_k^\dagger$  and  $\hat{\psi}_1, \dots, \hat{\psi}_k$  into new ones  $\hat{\phi}_1^\dagger, \dots, \hat{\phi}_k^\dagger$  and  $\hat{\phi}_1, \dots, \hat{\phi}_k$ . We can write this transformation with the help of a unitary matrix  $\hat{V} = (v_{ij})$  of order  $k$  as follows:

$$\hat{\phi}_i^\dagger = \sum_{j=1}^k v_{ij} \hat{\psi}_j^\dagger, \quad \hat{\phi}_i = \sum_{j=1}^k v_{ij}^* \hat{\psi}_j. \quad (8.24)$$

In this way, a set of  $k$  new states  $|\alpha_i\rangle = \hat{\phi}_i^\dagger |\text{vac}\rangle$  is defined as a linear combination of the states  $|\mathbf{r}_j, t_j\rangle$ . We may wonder now, what is the correlation function

$$G^{(k)'} = \langle \hat{\phi}_1^\dagger \dots \hat{\phi}_k^\dagger \hat{\phi}_k \dots \hat{\phi}_1 \rangle \quad (8.25)$$

that expresses the probability of finding an electron in each of the new states  $|\alpha_i\rangle$ . After we have passed the calculation of  $G^{(k)}$ , it will be no problem to evaluate  $G^{(k)'}$ . We just have to realize that relations analogous to (4.11) and (4.12) hold between the operators  $\hat{\phi}_i, \hat{\phi}_i^\dagger$  and  $\hat{a}(p), \hat{a}^\dagger(p)$ , which can be obtained by substituting Eqs. (4.11) and (4.12) into

Eq. (8.24):

$$\hat{\phi}_i = \sum_{j=1}^k v_{ij}^* \hat{\psi}_j = \sum_{j=1}^k v_{ij}^* \sum_p K_i(p) \hat{a}(p) = \sum_p K'_i(p) \hat{a}(p) \quad (8.26)$$

(we write the formula for  $\hat{\phi}_i$  only; the Hermite conjugate of Eq. (8.26) holds for  $\hat{\phi}_i^\dagger$ ). Now we can repeat the calculation from Chapter 6 completely, the only difference being the prime over  $K(p)$ . The result will be therefore according to Eq. (6.28)

$$G^{(k)'} = \det \hat{\Gamma}' = \begin{vmatrix} \Gamma'_{11} & \Gamma'_{12} & \cdots & \Gamma'_{1k} \\ \Gamma'_{21} & \Gamma'_{22} & \cdots & \Gamma'_{2k} \\ \vdots & \vdots & & \vdots \\ \Gamma'_{k1} & \Gamma'_{k2} & \cdots & \Gamma'_{kk} \end{vmatrix}, \quad (8.27)$$

where

$$\Gamma'_{ij} = \langle \hat{\phi}_i^\dagger \hat{\phi}_j \rangle \quad (8.28)$$

is the cross-correlation function corresponding to the states  $|\alpha_i\rangle$  and  $|\alpha_j\rangle$ .

Let us examine the matrix  $\hat{\Gamma}'$  in more detail. Substituting the operators (8.24) into Eq. (8.28), we get

$$\Gamma'_{ij} = \langle \hat{\phi}_i^\dagger \hat{\phi}_j \rangle = \left\langle \sum_{l=1}^k v_{il} \hat{\psi}_l^\dagger \sum_{m=1}^k v_{jm}^* \hat{\psi}_m \right\rangle = \sum_{l=1}^k \sum_{m=1}^k v_{il} \langle \hat{\psi}_l^\dagger \hat{\psi}_m \rangle v_{jm}^* = \sum_{l=1}^k \sum_{m=1}^k v_{il} \Gamma_{lm} v_{jm}^*. \quad (8.29)$$

This equation shows that it holds  $\hat{\Gamma}' = \hat{V} \hat{\Gamma} \hat{V}^\dagger$ , that is, the matrix  $\hat{\Gamma}'$  can be obtained from  $\hat{\Gamma}$  by a unitary transformation  $\hat{V}$ . This has an interesting consequence: The determinants of the matrices  $\hat{\Gamma}'$  and  $\hat{\Gamma}$ , and hence also the two correlation functions of the  $k$ th order  $G^{(k)'}$ ,  $G^{(k)}$  are equal to each other,

$$G^{(k)'} = \langle \hat{\phi}_1^\dagger \cdots \hat{\phi}_k^\dagger \hat{\phi}_k \cdots \hat{\phi}_1 \rangle = \langle \hat{\psi}_1^\dagger \cdots \hat{\psi}_k^\dagger \hat{\psi}_k \cdots \hat{\psi}_1 \rangle = G^{(k)}. \quad (8.30)$$

At this place we compare the situation with the case of bosons. According to Mandel and Wolf [1], p. 428, Eq. (8.4-2b), the  $k$ -photon correlation function for thermal light has the form

$$O^{(k)} = \sum_{P \in \mathcal{P}} \langle \hat{F}_1^\dagger \hat{F}_{P(1)} \rangle \langle \hat{F}_2^\dagger \hat{F}_{P(2)} \rangle \cdots \langle \hat{F}_k^\dagger \hat{F}_{P(k)} \rangle, \quad (8.31)$$

where  $\hat{F}_1^\dagger$  and  $\hat{F}$  are the photon creation and annihilation operators and  $P$  denotes again a permutation of the numbers  $1, \dots, k$ . Comparing this with the electron correlation function written now in a similar manner

$$G^{(k)} = \sum_{P \in \mathcal{P}} \text{sign}(P) \langle \hat{\psi}_1^\dagger \hat{\psi}_{P(1)} \rangle \langle \hat{\psi}_2^\dagger \hat{\psi}_{P(2)} \rangle \cdots \langle \hat{\psi}_k^\dagger \hat{\psi}_{P(k)} \rangle, \quad (8.32)$$



we see that the photon correlation function has a very similar structure as the electron one, but with one fundamental difference — in the case of photons, all the terms enter the sum with a positive sign, irrespective to what the sign of the corresponding permutation is, while in the case of electrons the terms enter the sum with different signs according to the sign of the corresponding permutation. Now we can make the same procedure for photons as described above for electrons, i.e., define a new set of field operators  $\hat{R}_1^\dagger, \dots, \hat{R}_k^\dagger$  and  $\hat{R}_1, \dots, \hat{R}_k$  using a unitary transformation analogous to (8.24). Then the new correlation function  $\langle \hat{R}_1^\dagger \cdots \hat{R}_k^\dagger \hat{R}_k \cdots \hat{R}_1 \rangle$  will again have the form analogous to (8.31), now of course with new cross-correlation functions  $\langle \hat{R}_i^\dagger \hat{R}_{P(i)} \rangle$ . However, this new correlation function is no more equal to the original one (8.31). This was true only for electrons because of the nice properties of determinants, namely, the property of invariance under a unitary transformation. This invariance is possible because the various products enter the sum with various signs, which does not happen for photons.

Thus we have found an invariant of electron correlation functions that is no invariant in the case of photons. This seems to be one of the few things in which the electrons can be considered to be more interesting than photons from the theoretical point of view.

## 8.4 Modes of the field at the detectors

Now, let us choose the coefficients  $v_{ij}$  in such a way that the matrix  $\Gamma'$  is diagonal. Then the correlation function  $G^{(k)'}$  corresponding to the new states  $|\alpha_i\rangle$  reduces simply to the product of the one-electron correlation functions:

$$G_{1,\dots,k}^{(k)'} = \det(\Gamma') = \Gamma'_{11} \cdots \Gamma'_{kk} = G_1^{(1)'} \cdots G_k^{(1)'}. \quad (8.33)$$

This means, however, that the events of finding an electron at the different states  $|\alpha_i\rangle$  are totally independent, or, in other words, that the field at the different states  $|\alpha_i\rangle$  is mutually completely incoherent. In this way, we have found some “modes” of the correlation function. Indeed, each state  $|\alpha_i\rangle$  is a linear combination of the states  $|\mathbf{r}_j, t_j\rangle$  and the field at different states  $|\mathbf{r}_j, t_j\rangle, |\mathbf{r}_l, t_l\rangle$  can be mutually coherent, but the field in any two different new states  $|\alpha_j\rangle, |\alpha_l\rangle$  is now incoherent. This situation is very similar to the problem of coupled harmonic oscillators, where the oscillations of the individual oscillators are also coupled but the different modes no more.

It would be very interesting to see what would happen if the spatial density of the points  $(\mathbf{r}_i, t_i)$  went to infinity. I think that then the states  $|\alpha_i\rangle$  would become identical with the modes  $|p\rangle$  of the field (see Chapter 5) and that the diagonal elements of the matrix  $\Gamma'$  would express the “intensity” of these modes in the electron field.

## Chapter 9

# Two-electron correlation function for field-emission electrons

After we have calculated the correlation function of any order for chaotic electrons and derived important inequalities holding for correlation functions of different orders, it is time to concentrate on more practical problems related to experiment. From the experimental point of view, the three-, four- and more-electron correlation functions are of no use because nowadays it is still completely impossible to make any measurements of them. Then the one- and two-electron correlation functions remain. The one-electron correlation function is simply the probability of finding an electron at a certain place and is in fact measured daily in electron interferometers. This function provides no information about the correlational properties of the electrons arising from their indistinguishableness. Therefore it will be reasonable to concentrate on the two-electron correlation function in the following and to describe its basic properties for an experimental setup with a field-emission source.

Substituting  $k = 2$  into Eq. (6.28) and to Eq. (6.30), we obtain the following expressions for the two-electron correlation function at the points  $(\mathbf{r}_1, t_1), (\mathbf{r}_2, t_2)$ :

$$G_{12}^{(2)} = \Gamma_{11}\Gamma_{22} - \Gamma_{12}\Gamma_{21} = \Gamma_{11}\Gamma_{22}(1 - |\gamma_{12}|^2). \quad (9.1)$$

It is very useful to define the normalized two-electron correlation function by dividing  $G_{12}^{(2)}$  by the product of both the one-electron correlation functions  $G_i^{(1)} = \Gamma_{ii}$ :

$$g_{12}^{(2)} = \frac{G_{12}^{(2)}}{G_1^{(1)}G_2^{(1)}} = \frac{\Gamma_{11}\Gamma_{22} - \Gamma_{12}\Gamma_{21}}{\Gamma_{11}\Gamma_{22}} = 1 - |\gamma_{12}|^2. \quad (9.2)$$

This function characterizes the correlational properties of the electron field in a very convenient way. If we had completely uncorrelated particles, the probability of detecting one of them at the point  $(\mathbf{r}_1, t_1)$  and another one at the point  $(\mathbf{r}_2, t_2)$  would factorize into the product of the individual probabilities. Therefore the normalized two-particle correlation function would be equal to unity for any configuration of the two points  $(\mathbf{r}_1, t_1)$  and  $(\mathbf{r}_2, t_2)$ . So we see that the difference between the normalized one-electron correlation function and unity is a measure of the electron correlations. This difference is negative,

which corresponds to fermionic nature of electrons, and equal to  $-|\gamma_{12}|^2$ , suggesting that the correlation grows with the growing mutual coherence of the electron field at the points  $(\mathbf{r}_1, t_1)$  and  $(\mathbf{r}_2, t_2)$ .

To evaluate the correlation function  $g_{12}^{(2)}$ , we will need according to Eq. (9.2) the cross-correlation function  $\Gamma_{12}$  that can be calculated with the help of Eq. (6.27). To be able to perform the summation over all the modes in this equation, we must first specify how these modes in fact look like.

## 9.1 Modes of the field-emission electron field

Until now, we have not specified very much what we exactly mean by the modes  $|p\rangle$ , even though we made a few assumptions about them only. We assumed that the modes form an orthonormal and complete system in the one-electron Hilbert space, that different modes are mutually uncorrelated and that the set of the modes is discrete.

The first two assumptions were important for the calculation performed in Chapter 6. Indeed, during the calculation we used the anticommutation relations (4.13) for the creation and annihilation operators  $\hat{a}^\dagger(p)$ ,  $\hat{a}(p)$  of the modes  $p$ . The first relation in (4.13) is closely connected with the orthonormality of the set of the modes. In fact, it generally holds

$$\hat{a}^\dagger(p)\hat{a}(p') + \hat{a}(p')\hat{a}^\dagger(p) = \langle p'|p\rangle, \quad (9.3)$$

so if the different modes are not orthogonal, the first anticommutation relation in (4.13) does not hold. At the same time, we used the relations (4.11) and (4.12) to express the field operators  $\hat{\psi}^\dagger(\mathbf{r}, t)$ ,  $\hat{\psi}(\mathbf{r}, t)$  in terms of  $\hat{a}^\dagger(p)$ ,  $\hat{a}(p)$ . If this should be possible, it is necessary that the states  $|p\rangle$  form a complete set. The third important statement was that different modes are mutually uncorrelated. The fourth assumption that the set of modes  $p$  is discrete was used only during the calculation and can be completely relaxed (this has been already explained at the argumentation leading to the approximation (6.8)). Physically there is totally no problem in going over to a continuous set of modes.

We would like to emphasize that our assumptions are not fundamental; in principle, we could describe the electron field using any modes. But, because of their key role in the performed calculation, we have to take the first three of them into account when applying our results on a specific physical situation.

A typical set of modes satisfying the first two assumptions is the set of all momenta eigenstates. From the theoretical point of view, this situation is interesting. However, in an electron interference experiment with a field-emission source we hardly deal with plane-wave electron states, but rather with some kind of spherical-wave states. Therefore we will not concentrate on the momenta eigenstates but will look for the set of modes that correspond better to the physical reality.

To describe the basic correlation properties of a field-emission source, we will use the following simple model of the source: we will suppose that each point of its surface emits spherical waves, that emissions from different points are independent of each other, and that the energy distribution of the emitted electrons is the same for all the points of the source.

This suggests that monochromatic spherical waves with their origin at the source surface could be the modes we are looking for. However, the first and second assumptions mentioned above are not satisfied for them. Even though two spherical waves with different energies are always orthogonal, two waves with equal energies issuing from different points need not to be. The assumption about the completeness is even more corrupted: it is completely impossible to express an arbitrary state using just such a limited set of spherical waves. Fortunately, both of these problems can be solved, which we will show now.

We start with the first one. The solution is similar to the approximation (6.8). Very loosely speaking, there are relatively few pairs of plane waves that are not mutually orthogonal with respect to the number of pairs that are orthogonal. More precisely said, if a measure is defined on the set of the pairs of the plane waves that is finite for the set of all orthogonal pairs, then the measure of the set of all non-orthogonal pairs is equal to zero. In this way, the fact that some waves are not mutually orthogonal does not influence the result and we can apply the calculations from Chap. 6 also to the case of spherical waves. However, we give up that this argumentation is very heuristic and would deserve a strict mathematical proof. This may come in future, but for now we believe that it is correct.

The second problem with the assumption about the completeness can be solved quite easily. For the field operators  $\hat{\psi}^\dagger(\mathbf{r}, t)$  and  $\hat{\psi}(\mathbf{r}, t)$  the relations (4.11) and (4.12) hold. We can define the operators  $\hat{\phi}^\dagger(\mathbf{r}, t)$  and  $\hat{\phi}(\mathbf{r}, t)$  by relations similar to (4.11), (4.12), but omit some modes in the summation. This means that we do not sum over the complete set of the modes  $p$ , but over some subset  $P$  only:

$$\hat{\phi}^\dagger(\mathbf{r}, t) = \sum_{p \in P} K^*(\mathbf{r}, t|p) \hat{a}^\dagger(p), \quad \hat{\phi}(\mathbf{r}, t) = \sum_{p \in P} K(\mathbf{r}, t|p) \hat{a}(p). \quad (9.4)$$

Then it is clear that if all the occupied modes of the electron field belong to the subset  $P$ , the expectation value of any term composed of the operators  $\hat{\phi}^\dagger(\mathbf{r}, t)$ ,  $\hat{\phi}(\mathbf{r}, t)$  will be the same as that of the same term composed of  $\hat{\psi}^\dagger(\mathbf{r}, t)$ ,  $\hat{\psi}(\mathbf{r}, t)$ . Thus, concerning the correlation functions there is no difference between  $\hat{\phi}^\dagger(\mathbf{r}, t)$ ,  $\hat{\phi}(\mathbf{r}, t)$  and  $\hat{\psi}^\dagger(\mathbf{r}, t)$ ,  $\hat{\psi}(\mathbf{r}, t)$ , as long as we have included all the modes produced by the source into the sums in Eq. (9.4). This means also that by calculating the correlation functions using Eq. (6.26) or Eq. (6.27), the summation over the incomplete set of modes only, that is, over all the modes produced by the source, is completely correct.

So, we can consider spherical waves as the modes of the electron field. Each such wave  $|\mathbf{R}, E\rangle$  can be described by the following two parameters: the origin  $\mathbf{R}$  from which it propagates and the energy  $E$ . However, because of the simplification of notation, it will be more useful to use the physical quantity  $p = \sqrt{2mE}$  instead of energy<sup>1</sup>. We will call this quantity as “momentum”, even if it is not a momentum in the ordinary sense. This can be justified by the fact that in a large distance from the source there is not much difference between a spherical wave with energy  $p^2/(2m)$  and a plane wave with the magnitude of momentum equal to  $p$ . We will write the propagator corresponding to the spherical wave

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<sup>1</sup>please do not confuse this  $p$  with the notation of the modes of the field!

with momentum  $p$  coming from the point  $\mathbf{R}$  as follows,

$$K(\mathbf{r}, t | \mathbf{R}, p) = \frac{c}{|\mathbf{r} - \mathbf{R}|} \exp \left[ \frac{i}{\hbar} \left( p|\mathbf{r} - \mathbf{R}| - \frac{p^2}{2m}t \right) \right], \quad (9.5)$$

where  $c$  is a constant that is not important to us.

In our situation the summation over all modes  $q$  of the field that appears in Eq. (6.26) or (6.27) is equivalent to integration over all the momenta  $p$  of the spectrum and over all the points  $\mathbf{R}$  of the source. As the energy distribution is the same for all the points of the source, the mode spectrum function  $f(q)$  factorizes into the product of the normalized momentum distribution  $F(p)$  and the source emission intensity  $J(\mathbf{R})$ :

$$f(q) = \frac{1}{\mathcal{I}} J(\mathbf{R})F(p). \quad (9.6)$$

Here the total source emission intensity  $\mathcal{I}$  appears in the denominator to ensure the proper normalization of the probability distribution  $f(q)$ .

Using the formula (6.27) for the cross-correlation function  $\Gamma_{12} = \langle \hat{\psi}^\dagger(\mathbf{r}_1, t_1) \hat{\psi}(\mathbf{r}_2, t_2) \rangle$  at the points  $(\mathbf{r}_1, t_1)$ ,  $(\mathbf{r}_2, t_2)$ , Eq. (9.6) and the spherical wave propagator (9.5), we can write the following expression for  $\Gamma_{12}$ :

$$\Gamma_{12} = \frac{N|c|^2}{\mathcal{I}} \int_0^\infty dp \int_V dV \frac{F(p)J(\mathbf{R})}{|\mathbf{r}_1 - \mathbf{R}||\mathbf{r}_2 - \mathbf{R}|} \times \exp \left\{ \frac{i}{\hbar} \left[ (|\mathbf{r}_2 - \mathbf{R}| - |\mathbf{r}_1 - \mathbf{R}|)p - \frac{p^2}{2m}(t_2 - t_1) \right] \right\}. \quad (9.7)$$

The second integral runs over the volume  $V$  of the source. As in our model we assumed that the source emits only from its surface, the intensity distribution  $J(\mathbf{R})$  should be understood to be nonzero on the surface of the source only where it has a Dirac  $\delta$ -function-like singularity. This will be evident from an example that comes soon.

For the following calculations we will make a few approximations, all of which are completely acceptable for practical applications. First, it is clear that the size of the source that is a fraction of a micrometer is by many orders less than the distances  $r_1, r_2$  between the source and the detectors that are several decimeters. If we place the origin of coordinates to the center of the field emission tip, we can therefore neglect  $\mathbf{R}$  in the denominator of Eq. (9.7) with respect to  $\mathbf{r}_1$  and  $\mathbf{r}_2$  and take the whole fraction out of the integral. We cannot of course make such a rough approximation in the exponent of Eq. (9.7) because even small changes of  $\mathbf{R}$  can cause large changes of the exponent because of an extremely small wavelength of the electrons. However, an approximation can be still made. For this reason we decompose the vector  $\mathbf{u} = \mathbf{r} - \mathbf{R}$  into a sum of two vectors: the first one  $\mathbf{v}_\parallel$  parallel and the second one  $\mathbf{v}_\perp$  orthogonal to  $\mathbf{r}$ . If  $R = |\mathbf{R}| \ll |\mathbf{r}| = r$ , the magnitude of  $\mathbf{v}_\perp$  will be many orders less than the magnitude of  $\mathbf{v}_\parallel$  and because the two vectors are orthogonal to each other, the magnitude of  $\mathbf{r}_i - \mathbf{R}$  will be determined by the magnitude of  $\mathbf{v}_\parallel$  only (up to the first order in  $R/r$ ). As it holds

$$\mathbf{v}_\parallel = \left[ (\mathbf{r} - \mathbf{R}) \frac{\mathbf{r}}{r} \right] \frac{\mathbf{r}}{r} = \left( r - \frac{\mathbf{r}\mathbf{R}}{r} \right) \frac{\mathbf{r}}{r}, \quad (9.8)$$

we can write

$$|\mathbf{r} - \mathbf{R}| \approx |\mathbf{v}_{\parallel}| = r - \frac{\mathbf{r}\mathbf{R}}{r}. \quad (9.9)$$

The exponent in Eq. (9.7) then becomes

$$\frac{i}{\hbar} \left[ p(r_2 - r_1) - \frac{p^2}{2m}(t_2 - t_1) - p \left( \frac{\mathbf{r}_2}{r_2} - \frac{\mathbf{r}_1}{r_1} \right) \mathbf{R} \right] = \frac{i}{\hbar} \left[ pl - \frac{p^2}{2m}\tau - p\mathbf{u}\mathbf{R} \right], \quad (9.10)$$

where we have denoted  $\tau = t_2 - t_1$ ,  $l = r_2 - r_1$ ,  $\mathbf{u} = \mathbf{r}_2/r_2 - \mathbf{r}_1/r_1$ . With the help Eq. (9.10), we can write Eq. (9.7) in the following way:

$$\Gamma_{12} = \frac{N|c|^2}{|\mathbf{r}_1||\mathbf{r}_2|} \int_0^\infty F(p) A(\mathbf{u}, p) \exp \left[ \frac{i}{\hbar} \left( pl - \frac{p^2}{2m}\tau \right) \right] dp, \quad (9.11)$$

where the function  $A(\mathbf{u}, p)$  is defined as

$$A(\mathbf{u}, p) = \frac{1}{\mathcal{I}} \int_V J(\mathbf{R}) \exp \left( -\frac{i}{\hbar} p\mathbf{u}\mathbf{R} \right) dV. \quad (9.12)$$

To evaluate the complex degree of coherence  $\gamma_{12}$  of the electron field at the points  $(\mathbf{r}_1, t_1)$  and  $(\mathbf{r}_2, t_2)$ , we will need also the one-electron correlation functions  $G_1^{(1)}$  and  $G_2^{(1)}$ . Their evaluation is very simple: in analogy with Eq. (9.7) we get now

$$G_i^{(1)} = \Gamma_{ii} = \frac{N|c|^2}{\mathcal{I}} \int_0^\infty dp \int_V dV \frac{F(p)J(\mathbf{R})}{|\mathbf{r}_i - \mathbf{R}|^2} = \frac{N|c|^2}{r_i^2} \quad (9.13)$$

and the complex degree of coherence then becomes with the help of Eqs. (9.11) and (9.13)

$$\gamma_{12} = \frac{\Gamma_{12}}{\sqrt{\Gamma_{11}\Gamma_{22}}} = \int_0^\infty F(p) A(\mathbf{u}, p) \exp \left[ \frac{i}{\hbar} \left( pl - \frac{p^2}{2m}\tau \right) \right] dp. \quad (9.14)$$

The equation (9.13) shows that the one-electron correlation functions are inversely proportional to the square of the distance from the source, which could be expected.

## 9.2 Angular coherence function

From Eq. (9.12) it is clear that the function  $A(\mathbf{u}, p)$  depends on  $\mathbf{u}$  and does not depend on the distances of the detectors from the source and times of measurement at the detectors. Let us see what is in fact the physical meaning of  $\mathbf{u}$ . From the cosine theorem applied on the triangle with one vertex at the origin of coordinates and the other two ones at the detectors it follows that  $|\mathbf{u}| = \sqrt{2(1 - \cos \varphi)} = 2 \sin(\varphi/2)$ , where  $\varphi$  is the angle between the vectors  $\mathbf{r}_1$  and  $\mathbf{r}_2$ . For small angles it holds approximately  $|\mathbf{u}| = \varphi$ . The direction of the vector  $\mathbf{u}$  is perpendicular to the axis of the angle  $\varphi$ . This suggests that  $A(\mathbf{u}, p)$

describes the angular coherence properties of the electron field and therefore we will call it the angular coherence function.

The equation (9.12) shows that the function  $A(\mathbf{u}, p)$  is almost the three-dimensional Fourier transform of the intensity of radiation  $J(\mathbf{R})$ , the only difference being that the integration variable  $\mathbf{R}$  runs only over the source volume. If we define, however, the intensity of radiation also outside the source to be equal to zero,  $A(\mathbf{u}, p)$  becomes a true Fourier transform of the intensity of radiation of the source. In this way we have derived the far-zone form of the van Cittert — Zernike theorem for electrons that is known in classical as well as in quantum optics ([1], pages 188 — 191). This theorem states that the angular part of the cross-correlation function is the normalized Fourier transform of the intensity distribution of the source, which is precisely Eq. (9.12).

### 9.2.1 Gaussian distribution of the source intensity

To illustrate the properties of the angular coherence function, we will calculate it now for a source radiating from the  $xy$ -plane with a two-dimensional Gaussian distribution of emission intensity. The function  $J(\mathbf{R})$  will be then

$$J(\mathbf{R}) = \frac{\mathcal{I}}{2\pi D_1 D_2} \exp\left(-\frac{x^2}{2D_1^2} - \frac{y^2}{2D_2^2}\right) \delta(z). \quad (9.15)$$

Here  $D_1$  and  $D_2$  are characteristic sizes of the source in the  $x$  and  $y$  directions and the presence of the Dirac delta-function reflects the fact that the source radiates from a two-dimensional region only. The angular coherence function  $A(\mathbf{u}, p)$  becomes then according to Eq. (9.12)

$$A(\mathbf{u}, p) = \frac{1}{2\pi D_1 D_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(z) \exp\left[-\frac{x^2}{2D_1^2} - \frac{y^2}{2D_2^2} + \frac{ip}{\hbar}(u_x x + u_y y + u_z z)\right] dx dy dz \quad (9.16)$$

The evaluation of the triple integral is not difficult and yields

$$A(\mathbf{u}, p) = \exp\left[-\frac{(u_x)^2}{2\Phi_1^2} - \frac{(u_y)^2}{2\Phi_2^2}\right], \quad (9.17)$$

where  $\Phi_i = \hbar/(D_i p)$  are coherence angles corresponding to the  $x$  and  $y$  directions. The angular coherence function  $A(\mathbf{u}, p)$  is hence also Gaussian and its width in a particular direction ( $x$  or  $y$ ) is inversely proportional to the source extension in the same direction. For the case of a linear source we put  $D_2 = 0$  in Eq. (9.17). Then  $\Phi_2 \rightarrow \infty$  and the angular coherence function becomes

$$A(\mathbf{u}) = \exp\left[-\frac{(u_x)^2}{2\Phi_1^2}\right], \quad (9.18)$$

that is, depends no more on  $u_y$ . This means that even if the two detectors are placed very far from each other in the  $y$  direction, the angular coherence remains preserved.

The equation (9.17) shows that the angular coherence increases with a decreasing size of the source. This is not a property of the Gaussian intensity distribution only. It holds generally for pairs of functions that are Fourier transforms of one another that their “widths” are inversely proportional to each other with the proportionality factor of order of unity. If  $D$ ,  $|\mathbf{u}_{\max}|$  and  $\lambda = 2\pi\hbar/p$  denote the characteristic size of the source, the maximal  $|\mathbf{u}|$  for which the function  $A(\mathbf{u})$  does not vanish and the wavelength of the electrons, respectively, it holds approximately

$$\frac{p|\mathbf{u}_{\max}|D}{\hbar} \approx 2\pi \quad \Longrightarrow \quad |\mathbf{u}_{\max}| \approx \frac{2\pi\hbar}{pD} = \frac{\lambda}{D}. \quad (9.19)$$

The function  $A$  therefore vanishes for  $|\mathbf{u}_{\max}| > \lambda/D$  which results in nullifying of the cross correlation function and thus loss of coherence at the points  $(\mathbf{r}_1, t_1)$  and  $(\mathbf{r}_2, t_2)$ . If  $|\mathbf{u}_{\max}|$  is small, it can be identified with the coherence angle. Eq. (9.19) then shows that the coherence angle of a source with a characteristic size  $D$  is equal to  $\lambda/D$ . This corresponds again very well to the analogous situation in optics where a similar result is obtained.

As it is clear from Eqs. (9.12) and (9.19), the coherence angle and the angular coherence function depend on the momentum  $p$  of the electrons. For quasi-monochromatic electrons, however, this dependence can be usually neglected because of a narrow width  $\delta$  of the spectral function  $F(p)$ . To show it, let us see how much the term  $p\mathbf{u}\mathbf{R}/\hbar$  in the exponent of Eq. (9.12) changes with  $p$  for a given  $\mathbf{u}$  and  $\mathbf{R}$ . From the definition of  $\mathbf{u}$  it follows that  $|\mathbf{u}| \leq 2$ , but from the experimental point of view,  $|\mathbf{u}|$  is limited by the emission angle of the source which is about a tenth of a radian. At the same time,  $R$  is limited by the characteristic size  $D$  of the source. Therefore

$$\left| \Delta \left( \frac{p\mathbf{u}\mathbf{R}}{\hbar} \right) \right| < \frac{0.1\delta D}{\hbar} = \frac{D}{10l_c}, \quad (9.20)$$

holds, where we have denoted  $l_c = \hbar/\delta$  is the coherence length of the electrons. For coherence length of about 60 nm (this corresponds to energy 2 keV and energy bandwidth 0.3 eV) and the source size  $D$  of about 50 nanometers, we see that the change of the term in the exponent of Eq. (9.12) is small compared to unity. This must then hold also for the whole angular coherence function, which will be used in the following.

### 9.3 Temporal and longitudinal coherence function

Now we come back to Eq. (9.11). Using the fact that the dependence of  $A(\mathbf{u}, p)$  on  $p$  can be neglected for quasi-monochromatic electrons, we can write the complex degree of coherence simply as a product

$$\gamma_{12} = A(\mathbf{u}, p_0) L(l, \tau), \quad (9.21)$$

where

$$L(l, \tau) = \int_0^\infty F(p) \exp \left\{ \frac{i}{\hbar} \left[ lp - \frac{p^2}{2m} \tau \right] \right\} dp. \quad (9.22)$$



The function  $L(l, \tau)$  depends on the distances of the detectors from the source and the times of the measurement and does not depend on the directions from the source to the detectors. This means that it describes the longitudinally-temporal coherence properties of the electron field, which is the reason why we will call it the temporal and longitudinal coherence function.

With the help of Eqs. (9.2) and (9.21), the normalized two-electron correlation function can be finally expressed as

$$g_{1,2}^{(2)} = g^{(2)}(\mathbf{u}, p_0, l, \tau) = 1 - |\gamma_{12}|^2 = 1 - |A(\mathbf{u}, p_0)|^2 |L(l, \tau)|^2. \quad (9.23)$$

## 9.4 Antibunching — the basic property of two-electron correlations

From Eq. (9.23) we see that if  $A(\mathbf{u}, p_0) = 0$ , that is, if the detectors are placed outside the coherence angle of the source, the normalized two-electron correlation function will be equal to unity for any  $\tau$  and  $l$ . This means that the electrons come to the detectors completely uncorrelated. This result can be expected.

To investigate the influence of the longitudinal and temporal coherence of the electron field on the two-electron correlation function, we will suppose now that both the detectors are situated well within the coherence angle of the source. This means that the square of the modulus  $|A(\mathbf{u})|^2$  of the angular coherence function is equal to unity and the coherence properties of the field at the two detectors are determined purely by the longitudinal and temporal coherence function  $L(l, \tau)$ . Then the normalized two-electron correlation function becomes according to Eqs. (9.22) and (9.23)

$$g^{(2)}(l, \tau) = 1 - \left| \int_0^\infty F(p) \exp \left\{ \frac{i}{\hbar} \left[ lp - \frac{p^2}{2m} \tau \right] \right\} dp \right|^2 \quad (9.24)$$

This equation shows that if both  $l$  and  $\tau$  are zero, the correlation function is also equal to zero because of the normalization of the function  $F(p)$ . This means that if the two detectors are equally distant from the source, it is impossible to detect an electron at both of them simultaneously. For the case that both the detectors are located at the same place, this statement is equivalent to the Pauli principle. However, for  $g_{1,2}^{(2)}$  to be equal to zero it is not necessary that the detectors are located at the same place. The equal distance from the source (of course together with the angular coherence condition) is sufficient because of the spherical-wave character of the the electron field emitted by the source. To see other properties of the correlation function (9.24), we will use an interesting analogy with a wave packet.

### 9.4.1 Analogy with a wave packet

Consider for a moment a single quantum particle with the mass  $m$  same as the electron (e.g. again an electron) and with the wave function  $a(p)$  in the momentum representation.

Then the coordinate wave function in the Schrödinger representation has the form

$$\psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} a(p) \exp \left[ \frac{i}{\hbar} \left( xp - \frac{p^2}{2m} t \right) \right] dp \quad (9.25)$$

and the probability density of finding the particle at the point  $x$  is equal to

$$|\psi(x, t)|^2 = \frac{1}{2\pi\hbar} \left| \int_{-\infty}^{\infty} a(p) \exp \left[ \frac{i}{\hbar} \left( xp - \frac{p^2}{2m} t \right) \right] dp \right|^2. \quad (9.26)$$

If we compare now this equation with the second term on the right-hand side of Eq. (9.24) (this is the term  $|\gamma_{12}|^2$ ), we see a strong similarity. If we could make the function  $s(p)$  equal to  $F(p)$ , this analogy would be complete and the value by which the normalized correlation function  $g_{1,2}^{(2)}$  differs from unity would be proportional to the probability density of finding the particle at the point  $x$  at the time  $t$ . We should not forget, however, that the function  $F(p)$  expresses the *probability* that an electron in our system has the momentum  $p$  while the function  $a(p)$  is the *probability amplitude* that the particle has the momentum  $p$ , so the functions  $F(p)$  and  $a(p)$  have different physical meaning as well as dimensions. Nevertheless, it is possible to imagine a particle with the momentum wave function  $a(p)$  proportional to the electron momentum distribution  $F(p)$ . Then the similarity between  $|\gamma_{12}|^2$  and  $|\psi(x, t)|^2$  can be very helpful for understanding the properties of the two-electron correlation function.

It is known that if the wave function  $a(p)$  is real, then the wave packet corresponding to this wave function is localized around the point  $x = 0$  at the time  $t = 0$ . As the function  $F(p)$  which corresponds to  $a(p)$  is real, a similar statement holds also for the function  $|\gamma_{12}|^2 = |\gamma(l, \tau)|^2$ . This has an important consequence: for  $l = 0$ , the function  $|\gamma(l, \tau)|^2$  will be nonzero around the point  $\tau = 0$  (i.e., when  $t_1 \approx t_2$ ), but for large  $\tau$  it will be equal to zero. Therefore the correlation function  $g_{1,2}^{(2)}$  will be reduced for  $t_1 \approx t_2$ , reach its minimum for  $t_1 = t_2$  and converge to unity for large time differences  $|t_2 - t_1|$ . This means that it is less likely to detect electrons at the two detectors short after one another than to detect them long after one another. We could also say that the electrons avoid coming to the detectors in pairs or groups, or “bunches”. This effect is called antibunching and is a counterpart of bunching of chaotic photons [22, 23].

If the wave packet (9.25) is quasi-monochromatic, which means that the function  $a(p)$  has a narrow peak around the point  $p_0$ , it has another interesting property: it moves as a whole with the group velocity  $v = p_0/m$ , changing its shape only a little. From the analogy we can thus expect that if the electrons are quasi-monochromatic, then also the correlation function  $g_{1,2}^{(2)}$  will “move” in time  $\tau$  along the axis  $l$  with the velocity  $v = p_0/m$ . So we can say that a longitudinal shift  $l$  of the detectors is equivalent to a temporal shift  $\tau = l/v$  of the times of measurement. So, similarly as it is impossible to detect two electrons at the two detectors simultaneously if  $l = 0$ , for  $l > 0$  it is impossible to detect

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<sup>2</sup>this means simply that  $g_{1,2}^{(2)}$  can be approximately expressed as a function of the variable  $l - v\tau$

an electron at the first detector at some moment of time and then detect an electron at the second detector  $l/v$  seconds later.

Even if the wave packet is quasi-monochromatic, its form changes a little with time — we say that it spreads. The spreading is slower for more monochromatic packets, but takes place always. It means that the length or extension of the packet becomes larger and larger during the time while its maximal height reduces at the same time (these two effects must take place simultaneously because the integral of  $|\psi(x, t)|^2$  over  $x$  remains equal to unity all the time). We can expect the same behaviour of our correlation function. Thus, for large time difference  $\tau$  of measurement on the detectors the term  $|\gamma_{12}|^2$  will not reach the value of unity for any detector configuration and therefore the correlation function  $g_{1,2}^{(2)} = 0$  will never reach the zero value. It will have only a minimum for  $l = v\tau$ . We can put it also in another way: if the detectors are placed in very different distances from the source ( $l$  is large), then the probability of detecting one electron at the first detector and another electron at the second detector  $\tau$  seconds later will be nonzero for any  $\tau$ . On the other hand, the interval of the time differences  $\tau$  for which the correlation function  $g_{1,2}^{(2)} = 0$  is reduced considerably will become longer than they were for  $l = 0$ .

### 9.4.2 Gaussian momentum distribution

To illustrate that the results obtained heuristically from the analogy between the two-electron correlation function and the coordinate probability distribution of a wave-packet state, we will consider now a specific momentum distribution. So, let the function  $F(p)$  have the Gaussian form with the mean value  $p_0$  and dispersion  $\delta^2$ :

$$F(p) = \frac{1}{\sqrt{2\pi}\delta} \exp\left[-\frac{(p-p_0)^2}{2\delta^2}\right]. \quad (9.27)$$

The temporal and longitudinal coherence function then becomes according to Eq. (9.22)

$$L(l, \tau) = \frac{1}{\sqrt{2\pi}\delta} \int_0^\infty \exp\left\{-\frac{(p-p_0)^2}{2\delta^2} + \frac{i}{\hbar} \left[lp - \frac{p^2}{2m}\tau\right]\right\} dp. \quad (9.28)$$

If the electrons are quasi-monochromatic, i.e., if  $\delta \ll p_0$ , we can extend the lower bound of the integral into minus infinity. The integral is then not difficult to solve and yields

$$L(l, \tau) = \frac{1}{\sqrt{1 - i\delta^2\tau/(m\hbar)}} \exp\left[\frac{i}{\hbar} \left(lp_0 - \frac{p_0^2\tau}{2m}\right) - \frac{1}{2\hbar^2} \frac{(l - v\tau)^2}{1/\delta^2 + i\tau/m\hbar}\right]. \quad (9.29)$$

When we divide the second term in the exponent into its real and imaginary parts, we can easily compute the correlation function  $g_{1,2}^{(2)} = g^{(2)}(l, \tau)$ :

$$g^{(2)}(l, \tau) = 1 - |L(l, \tau)|^2 = 1 - \frac{1}{\sqrt{1 + \delta^4\tau^2/(m^2\hbar^2)}} \exp\left[-\frac{(l - v\tau)^2}{\hbar^2/\delta^2 + \delta^2\tau^2/m^2}\right]. \quad (9.30)$$

This equation shows that the minimal values of the correlation function  $g^{(2)}(l, \tau)$  are situated around the point  $\tau = l/v$ , which corresponds to the “motion” of the correlation function. For large  $l$  the square root in Eq. (9.30) is significantly larger than unity around this point and therefore the minimal value of  $g_{1,2}^{(2)}$  is no more zero. At the same time, the term  $\delta^2 \tau^2 / m^2$  in the denominator of the exponent causes from the same reason the increase of the width of the negative peak of  $g^{(2)}(l, \tau)$ . The figures (9.1) and (9.2) show this behaviour clearly.

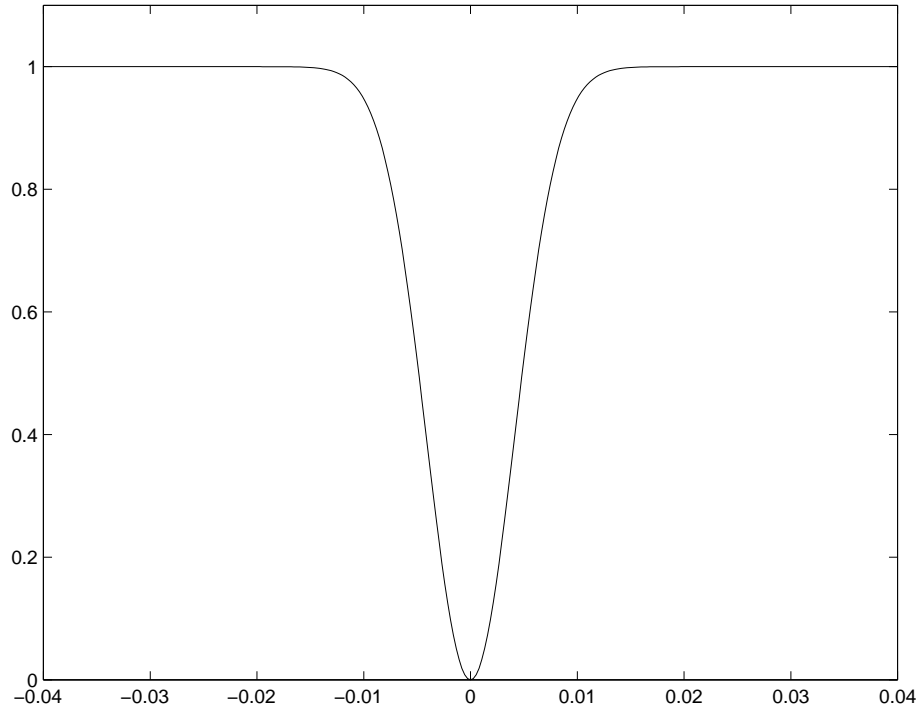


Figure 9.1: The normalized two-electron correlation function  $g^{(2)}(l, \tau)$  for  $l = 0$  as a function of the time difference  $\tau$  for 2 keV electrons with the energy bandwidth 0.3 eV and the Gaussian momentum distribution. The time  $\tau$  is expressed in picoseconds. The width of the peak shows an extremely short coherence time of the field-emission electrons of about  $10^{-14}$  sec.

## 9.5 Estimating the spatial and temporal extent of two-electron correlations

Let  $p_0$  and  $\delta^2$  denote the mean value and dispersion of the momentum distribution  $F(p)$ , respectively. Then we can find a normalized function  $h(u)$  centered around the point  $u = 0$  with the unity dispersion such that it holds

$$F(p) = \frac{1}{\delta} h\left(\frac{p - p_0}{\delta}\right). \quad (9.31)$$

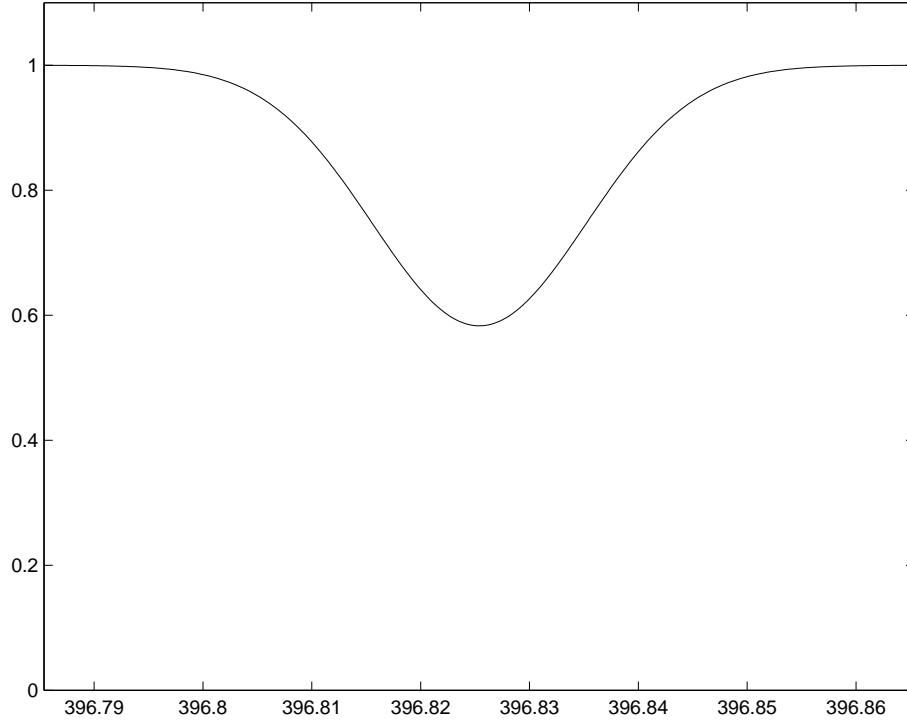


Figure 9.2: The normalized two-electron correlation function  $g^{(2)}(l, \tau)$  for  $l = 1$  centimeter as a function of the time difference  $\tau$  expressed in picoseconds. The other parameters are same as in Fig. 9.1. The spreading of the correlation function as well as the time shift of the minimum are obvious.

Now, let us express the temporal and longitudinal coherence function  $L(l, \tau)$  for  $\tau = 0$  according to Eq. (9.22) with the help of the function  $h(u)$ :

$$L(l, 0) = \frac{1}{\delta} \int_0^{\infty} h\left(\frac{p - p_0}{\delta}\right) \exp\left(\frac{ilp}{\hbar}\right) dp. \quad (9.32)$$

Using the substitution  $u = (p - p_0)/\delta$ , we can rewrite the integral as

$$L(l, 0) = \exp\left(\frac{ilp_0}{\hbar}\right) \int_{-\infty}^{\infty} h(u) \exp\left(\frac{iul\delta}{\hbar}\right) du. \quad (9.33)$$

In fact, the lower limit of the integral should be  $-p_0/\delta$ . However, if we define the function  $h(u)$  also for  $u < -p_0/\delta$  to be equal to zero, we can extend the lower limit to minus infinity. If we denote the integral as  $H(l\delta/\hbar)$ , the normalized correlation function becomes

$$g^{(2)}(l, 0) = 1 - \left| H\left(\frac{l\delta}{\hbar}\right) \right|^2. \quad (9.34)$$

We see that it is possible to express  $g^{(2)}(l, 0)$  as a function of  $l\delta/\hbar$ . This means that if we increase  $\delta$  while keeping the form of the function  $h(u)$ , the correlation function  $g^{(2)}(l, 0)$

shrinks in the  $l$  direction. The negative peak of  $g^{(2)}(l, 0)$  also shrinks and the correlation function is approaching unity as for uncorrelated particles. On the other hand, the width of the peak increases with reducing  $\delta$ . This can be expected: the more monochromatic are the electrons, the larger is the spatial extent where the equal-time correlations reach.

It is even possible to estimate the width of the peak. As the function  $h(u)$  has a unity dispersion, the dispersion of its Fourier transform  $H(l\delta/\hbar)$  will be also of order of unity<sup>3</sup>. Therefore the width of the function  $L(l, 0)$  will be of order of  $\hbar/\delta$ , which holds also for  $|L(l, 0)|^2$  and for the width of the negative peak of  $g^{(2)}(l, 0)$ . On the other hand,  $\hbar/\delta$  is nothing else than the coherence length  $l_c$  of the electrons. This means that the longitudinal extent of equal-time correlations in the electron field is given by the coherence length of the electrons, which is an important result.

Similarly, if we fix now  $l$  at the zero value and consider  $g^{(2)}(0, \tau)$  as a function of  $\tau$ , we arrive at the result that the negative peak in the correlation function has the width approximately equal to  $T_c = m\hbar/(p_0\delta)$ . This time  $T_c$  is obtained from the coherence length  $l_c$  by dividing by the mean velocity  $v = p_0/m$  of the electrons and can be identified with the coherence time of the electron field. This is another important result: if one electron comes to a detector, the probability that another one comes will be then reduced during the coherence time.

For typical field-emission electrons with the energy 2 keV and energy bandwidth of about 0.3 electronvolts, the momentum bandwidth is  $1.7 \times 10^{-27}$  kgm/s. The square root of the momentum dispersion is then equal to  $\delta = 5.5 \times 10^{-28}$  kgm/s, the coherence length 200 nm and the coherence time  $10^{-14}$  s. The coherence time can be also expressed as  $T_c = \hbar/\sigma_E$ , where  $\sigma_E^2$  is the energy dispersion. This is equivalent with the formula  $T_c = m\hbar/(p_0\delta)$ , however, because  $\sigma_E/\delta = p_0/m$ .

## 9.6 Time integral of the complex degree of coherence as a measure of the two-electron correlation

As we have seen, it is the difference of the normalized two-electron correlation function and unity (which is the similar correlation function for uncorrelated particles) what expresses the correlational properties of the electrons. However, as has been mentioned in Chap. 2, the resolution time of the coincidence experiment is about  $10^{-10}$  s, which makes it impossible to measure the correlation function  $g_{1,2}^{(2)}$  itself as a function of  $\tau$ . In fact, the measured statistics of time differences between the starting and stopping pulse is “blurred” compared to the correlation function. The result is that instead of a nice deep minimum with the width equal to the coherence time we observe a very shallow minimum with the width of the resolution time, and the area that is missing in the blurred minimum is the same as the area missing in the correlation function. Therefore it is reasonable to define the “antibunching intensity” (unfortunately, we were not able to find a better name for this quantity) as the  $\tau$ -integral of  $|\gamma(l, \tau)|^2$  over the experimental resolution time  $T_r$ . This

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<sup>3</sup>This holds especially if the function  $h(u)$  has a nice single peak. For “wild” functions this may not be true, but we suppose that the energy spectrum of the electrons is well-behaved.

integral is the measure of the two-electron correlational properties of the electron field that can be experimentally observed. However, as the interval where  $|\gamma(l, \tau)|^2$  is significantly different from zero is very short compared to the resolution time  $T_r$ , we can integrate over the whole time axis instead, so that

$$S = \int_{-\infty}^{\infty} |\gamma(l, \tau)|^2 d\tau = \int_{-\infty}^{\infty} |L(l, \tau)|^2 d\tau \quad (9.35)$$

(we still suppose that the angular coherence condition  $|A(\mathbf{u}, p_0)|^2 = 1$  is satisfied).

Let us try to express  $S$  in terms of the momentum distribution  $F(p)$  now. Substituting Eq. (9.22) into Eq. (9.35), we get for  $S$  the following triple integral:

$$S = \int_{-\infty}^{\infty} d\tau \int_0^{\infty} dp \int_0^{\infty} dq F(p)F(q) \exp \left\{ \frac{i}{\hbar} \left[ l(q-p) - \frac{(q^2-p^2)}{2m} \tau \right] \right\}. \quad (9.36)$$

Interchanging formally the order of integration and using the fact that

$$\int_{-\infty}^{\infty} \exp \left\{ -\frac{i}{\hbar} \frac{(q^2-p^2)}{2m} \tau \right\} d\tau = 4\pi\hbar m \delta(q^2-p^2) = \frac{2\pi\hbar m}{p} \delta(q-p) \quad (9.37)$$

(we used the fact that  $p, q$  are both positive), the integral (9.36) becomes

$$S = \pi\hbar m \int_0^{\infty} \int_0^{\infty} \frac{1}{p} F(p)F(q) \exp \left\{ \frac{i}{\hbar} [l(q-p)] \right\} \delta(q-p) dp dq = \pi\hbar m \int_0^{\infty} \frac{1}{p} [F(p)]^2 dp. \quad (9.38)$$

This equation shows an important result: the antibunching intensity  $S$  does not depend on the longitudinal separation of the detectors. This reflects the fact we have already pointed out — the reducing of the height of the function  $|\gamma(l, \tau)|^2$  takes place together with its spreading, similarly as for  $|\psi(x, t)|^2$  in the case of the wave packet. Therefore it does not matter if the two detectors are placed equally far from the source or not — the signal measured in the experiment remains preserved. This can make the measurement a little bit simpler in the way that it is not necessary to place both the detectors precisely into one plane perpendicular to the optical axis of the experiment. Preserving of the antibunching intensity can be seen in figures 9.1 and 9.2.

Using the function  $h(u)$  defined above, we can express the antibunching intensity with the help of the square root of the momentum dispersion  $\delta$  in the following way:

$$S = \frac{2\pi\hbar m}{\delta} \int_{-\infty}^{\infty} \frac{1}{p_0 + u\delta} [h(u)]^2 du \approx \frac{2\pi\hbar m}{p_0\delta} \int_{-\infty}^{\infty} [h(u)]^2 du \quad (9.39)$$

We have taken advantage of the quasi-monochromaticity of the electrons again. Eq. (9.39) shows that the antibunching intensity is inversely proportional  $\delta$ . As the function  $h(p)$  has a unity dispersion, the integral  $\int_{-\infty}^{\infty} [h(u)]^2 du$  is also of order of unity. Then we can conclude that the antibunching intensity  $S$  is equal to the coherence time  $T_c = m\hbar/(p_0\delta)$  up to a multiplicative constant of order of unity.

## 9.7 Feasibility of observing electron antibunching

The fact that the coherence time of the field-emission electrons is as short as  $10^{-14}$  s makes any correlation measurement extremely difficult. As the resolution time exceeds the coherence time at least by three orders of magnitude, the relative signal even in the optimal case cannot exceed the value of 1/1000. Therefore a special attention must be paid to many effects that would be completely unimportant in other types of experiments. Moreover, in a coincidence experiment it is necessary to measure for a long time to obtain a reliable statistics of the time differences because the number of counts fluctuates [24, 25].

The situation with electrons in the condensed matter is easier. The temporal technology enables to make smart structures such as a two-dimensional electron gas in which the quantum nature of the electrons manifests itself very strongly. It is even possible to make beam-splitters and similar elements for electrons that are hardly thinkable for free electrons. Therefore it is no wonder that more-electron correlations have been observed already several times in the recent few years (see [6, 7, 8, 9]). If we compare this with the extreme difficulty of observing correlations of electrons in the vacuum, we can ask if it has any sense at all to make experiments with the free electrons when it is possible to measure the same correlations in the condensed matter. Indeed, I think that it definitely has sense. One of the reasons is that even if the electrons in condensed matter behave as free particles in some aspects, in fact they are quite different from the free electrons. Therefore we can say that what is observed in the condensed matter are not electron correlations, but rather quasi-electron correlations, which is a motivation for trying to observe also correlations of true free electrons. Another challenge can be paradoxically the difficulty of the measurement itself.

I believe that in spite of the problems I have just described, it will be possible to make a successful experiment demonstrating the fermionic nature of the electrons via electron antibunching. I also hope that it will be the university in Tübingen where it will be managed for the first time.



# Chapter 10

## Two-electron correlation function in a real experiment

In the previous chapter we considered a very simple situation where there was just the electron source and two detectors. In a real experimental setup this configuration would not be possible. For example, from technical reasons the separation of the detectors that are supposed to detect individual electrons must be at least a few millimeters. Without a magnifying electrostatic quadrupole it would never be possible to illuminate both the detectors coherently, that is, to place them within the coherence angle of the source. The usage of the quadrupoles enlarges the coherently illuminated area by several orders of magnitude which makes it possible to place both the detectors within it. In the last chapter we also have not taken the electrostatic field of the source into account. Of course, any electromagnetic field (e.g. of the quadrupole or the field-emission tip) acting on the electrons changes the probability amplitude that an electron comes from the source to the detector, i.e., the propagator. Therefore a proper analysis of a correlation experiment should include calculating the electron propagators corresponding to the experimental setup, which is not the aim of this dissertation, however. Fortunately, even though the electron wavefronts are deformed under the influence of electromagnetic fields, the coherence time does not change and after the electrons pass the regions with the field, also the coherence length restores to its original value. Therefore we cannot expect any drastic changes in the observed phenomena with respect to the results from the previous chapter, and the main properties of correlation functions will remain preserved. Nevertheless, there is one element that may cause also considerable changes in the electron correlations — the electrostatic biprism. As in the experimental setup with the biprism there are two ways how an electron can come into either detector, interesting interference effects are to be expected. We will try to describe some of them in the following.

## 10.1 Correlation function in an electron biprism interferometer

In the experiment that we are going to describe there are basically the following elements: the electron source, the electron biprism, the Wien filter and two electron detectors. We will suppose that both the detectors are situated in the interference region of our interferometer, i.e., in the region where the two partial electron waves overlap and can interfere. Then there are two classical paths along which an electron can come from the source to any of the two detectors — it can move either left or right from the biprism filament. Figure 10.1 shows the experimental setup including the paths schematically. As we do not distinguish if an electron moves left or right in the experiment, the probability amplitude that an electron in the mode  $q$  arrives into either detectors (which is simply the propagator  $K_i(q)$ ) can be calculated as the sum of the two corresponding amplitudes:

$$K_i(q) = K_{i1}(q) + K_{i2}(q). \quad (10.1)$$

So, the propagators  $K_{i1}(p)$   $K_{i2}(p)$  correspond to the left and right paths from the source to the  $i$ th detector, respectively. They can be generally written in the form

$$K_{iu}(q) = A_i e^{i\varphi_{iu}(q)}, \quad (10.2)$$

where  $A_i$  and  $\varphi_{iu}$  is the amplitude and phase of the propagator  $K_{iu}(p)$ , respectively. As the amplitude  $A_i$  is given mainly by the distance between the source and the detector (see Eq. (9.5)), we can assume that it is the same for both the propagators  $K_{i1}(q)$  and  $K_{i2}(q)$ . The semiclassical approximation can be used successfully for evaluating the phases  $\varphi_{iu}(q)$  because the electron wavelength is by many orders less than the size of any element in the experiment (the wavelength of 2 keV electrons is about  $3 \times 10^{-11}$  meters only). If we denote the classical path going from the source to the  $i$ th detector via the  $u$ th path ( $u$  is equal to 1 or 2 for paths left or right from the biprism, respectively, see Fig. 10.1) as  $[iu]$ , the phase  $\varphi_{iu}(q)$  can be then calculated simply as the classical action corresponding to the path  $[iu]$  with the the energy  $E$  of the mode  $q$ , divided by the Planck constant  $\hbar$ .

Now, to evaluate the cross-correlation function we will go back to Eq. (6.27). Using Eq. (10.1) we obtain

$$\begin{aligned} \Gamma_{12} &= N \sum_q f(q) K_1^*(q) K_2(q) = N \sum_q f(q) [K_{11}^*(q) + K_{12}^*(q)] [K_{21}(q) + K_{22}(q)] \\ &= \Gamma_{1121} + \Gamma_{1122} + \Gamma_{1221} + \Gamma_{1222}, \end{aligned} \quad (10.3)$$

where

$$\Gamma_{1u2v} = N \sum_q f(q) K_{1u}^*(q) K_{2v}(q) = N A_1 A_2 \sum_q f(q) \exp \{i[\varphi_{2v}(q) - \varphi_{1u}(q)]\}. \quad (10.4)$$

To understand the result (10.3) better, we should realize that the event that two electrons come from the source to the detectors can happen by one of the following four possibilities (see figure 10.4):

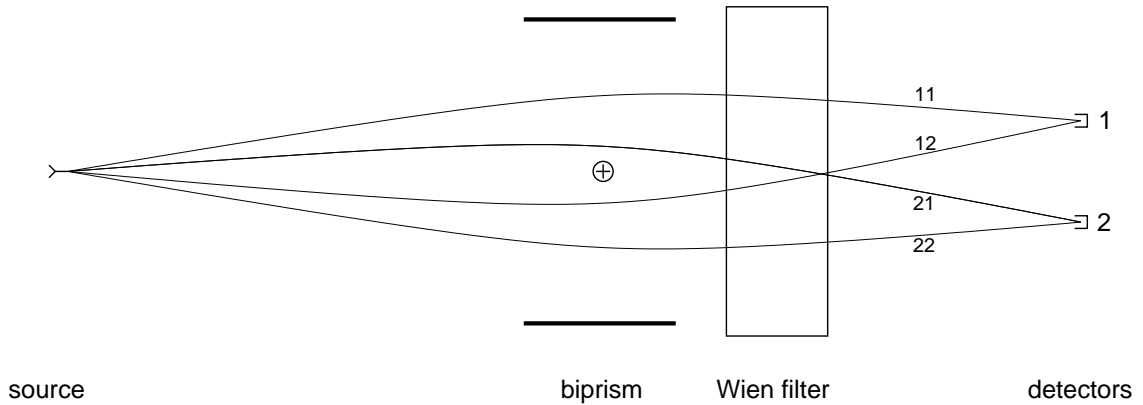


Figure 10.1: Scheme of the electron biprism interferometer with the four paths going from the source to the detectors 1 and 2. As the electrons are attracted by the positively charged filament, two classical paths for each detector exist.

1. both electrons take a path left from the biprism,
2. the electron that comes to detector 1 takes a path left from the biprism, and the electron that comes to detector 2 takes a path right from the biprism
3. the electron that comes to detector 1 takes a path right from the biprism, and the electron that comes to detector 2 takes a path left from the biprism,
4. both electrons take a path right from the biprism.

The cross-correlation functions corresponding to these four possibilities are evidently  $\Gamma_{1121}$ ,  $\Gamma_{1122}$ ,  $\Gamma_{1221}$ ,  $\Gamma_{1222}$ , respectively. As we do not distinguish between the four possibilities in the experiment, all of them interfere and therefore the total cross-correlation function is equal to the sum of the cross-correlation functions corresponding to all the possibilities.

The one-electron correlation functions  $G_i^{(1)} = \Gamma_{ii}$  can be evaluated easily again using Eq. (10.1) and Eq. (6.27):

$$\begin{aligned}
 G_i^{(1)} &= N \sum_q f(q) |K_i(q)|^2 = N \sum_q f(q) |K_{i1}(q) + K_{i2}(q)|^2 \\
 &= NA_i^2 \sum_q f(q) \{1 + \cos[\varphi_{i1}(q) - \varphi_{i2}(q)]\} \quad (10.5)
 \end{aligned}$$

If we define the complex degree of coherence corresponding to paths  $[iu]$  and  $[jv]$  as

$$\gamma_{iujv} = \sum_q f(q) \exp\{i[\varphi_{jv}(q) - \varphi_{iu}(q)]\}, \quad (10.6)$$

we can express the one-electron correlation functions, the complex degree of coherence and the normalized two-electron correlation function with the help of Eqs. (10.3) — (10.5) as

$$G_i^{(1)} = NA_i^2 (1 + \text{Re } \gamma_{i1i2}), \quad (10.7)$$

$$\gamma_{12} = \frac{|\gamma_{1121} + \gamma_{1122} + \gamma_{1221} + \gamma_{1222}|^2}{4(1 + \operatorname{Re} \gamma_{1112})(1 + \operatorname{Re} \gamma_{2122})}, \quad (10.8)$$

and

$$g_{1,2}^{(2)} = 1 - \frac{|\gamma_{1121} + \gamma_{1122} + \gamma_{1221} + \gamma_{1222}|^2}{4(1 + \operatorname{Re} \gamma_{1112})(1 + \operatorname{Re} \gamma_{2122})}. \quad (10.9)$$

Here  $\operatorname{Re}$  denotes the real part of a complex number. The equation (10.7) shows that the probability of detecting an electron at the  $i$ th detector oscillates if the phase of the complex degree of coherence  $\gamma_{i1i2}$  changes. This reflects the existence of interference fringes in the interference region. If the amplitude of  $\gamma_{i1i2}$  is small, which corresponds to a small angular or longitudinal coherence between the beams going to the  $i$ th detector left and right from the biprism filament, then the oscillations become small — the fringe contrast reduces.

### 10.1.1 Antibunching intensity in interferometer

We will calculate now the integral of the square of the absolute value of the complex degree of coherence  $|\gamma_{12}|^2$  over  $\tau$  in a similar way as we did in the previous chapter. For simplicity we will suppose that there is a complete angular coherence for any pair of paths  $[iu]$ ,  $[jv]$ . Then we can again use only the longitudinal and temporal coherence function for evaluating the complex degree of coherence, and, exactly as in the previous chapter, instead of summation over the modes  $q$  use integration over the momentum  $p$ . In this way the antibunching intensity with the help of Eq. (10.8) becomes

$$S = \int_{-\infty}^{\infty} |\gamma_{12}|^2 d\tau = \frac{\int_{-\infty}^{\infty} |\gamma_{1121} + \gamma_{1122} + \gamma_{1221} + \gamma_{1222}|^2 d\tau}{4(1 + \operatorname{Re} \gamma_{1112})(1 + \operatorname{Re} \gamma_{2122})}. \quad (10.10)$$

We have taken advantage of the fact that the denominator of the fraction (10.8) does not depend on  $\tau$ . For the numerator  $I$  of the fraction (10.10) we then get

$$I = \int_{-\infty}^{\infty} |\gamma_{1121} + \gamma_{1122} + \gamma_{1221} + \gamma_{1222}|^2 d\tau = \sum_{u,v,u',v'=1}^2 I_{uvu'v'}, \quad (10.11)$$

where

$$I_{uvu'v'} = \int_{-\infty}^{\infty} \gamma_{1u2v}^* \gamma_{1u'2v'} d\tau. \quad (10.12)$$

We introduce also the phase factor  $\phi_{iu}(p)$  as the time-independent part of the phase  $\varphi_{iu}(p)$ , so that

$$\varphi_{iu}(p) = \phi_{iu}(p) - \frac{p^2}{2m\hbar} t_i. \quad (10.13)$$

Then the integral  $I_{uvu'v'}$  can be expressed with the help of the integral form of Eq. (10.6) (comp. Eq. (9.14)) as

$$I_{uvu'v'} = \int_{-\infty}^{\infty} d\tau \int_0^{\infty} dp \int_0^{\infty} dq F(p)F(q) \times \exp \left\{ i \left[ \phi_{1u}(p) - \phi_{2v}(p) - \phi_{1u'}(q) + \phi_{2v'}(q) \right] - \frac{i}{\hbar} \frac{(q^2 - p^2)}{2m} \tau \right\}. \quad (10.14)$$

Now we proceed in a complete analogy to the previous chapter, Eqs. (9.37) and (9.38) to get

$$I_{uvu'v'} = \pi \hbar m \int_0^{\infty} \frac{1}{p} [F(p)]^2 \exp \left\{ i \left[ \phi_{1u}(p) - \phi_{1u'}(p) - \phi_{2v}(p) + \phi_{2v'}(p) \right] \right\} dp. \quad (10.15)$$

It is not difficult to show that

$$\sum_{u,v,u',v'=1}^2 \exp \left\{ i \left[ \phi_{1u}(p) - \phi_{1u'}(p) - \phi_{2v}(p) + \phi_{2v'}(p) \right] \right\} = 4 [1 + \cos \Delta\phi_{1112}(p)] [1 + \cos \Delta\phi_{2122}(p)], \quad (10.16)$$

where  $\Delta\phi_{iujv}(p) = \phi_{iu}(p) - \phi_{jv}(p)$ . Then the numerator of the fraction (10.10) becomes

$$I = 4\pi \hbar m \int_0^{\infty} \frac{1}{p} [F(p)]^2 [1 + \cos \Delta\phi_{1112}(p)] [1 + \cos \Delta\phi_{2122}(p)] dp. \quad (10.17)$$

The real part of the complex degrees of coherence corresponding to different paths along the biprism filament leading to the  $i$ th detector in the denominator of the fraction (10.10) can be expressed as

$$\text{Re } \gamma_{i1i2} = \int_0^{\infty} F(p) [1 + \cos \Delta\phi_{i1i2}(p)] dp. \quad (10.18)$$

The time-dependent parts of the phases have canceled now, so we deal with the phases  $\phi_{i1i2}(p)$  only. With the help of Eqs. (10.17), (10.18) and (10.10) we express finally the antibunching intensity in the interferometer:

$$S = \pi \hbar m \frac{\int_0^{\infty} \frac{1}{p} [F(p)]^2 [1 + \cos \Delta\phi_{1112}(p)] [1 + \cos \Delta\phi_{2122}(p)] dp}{\int_0^{\infty} F(p) [1 + \cos \Delta\phi_{1112}(p)] dp \int_0^{\infty} F(p) [1 + \cos \Delta\phi_{2122}(p)] dp}. \quad (10.19)$$

Comparing this result with  $S$  without interferometer according to (9.38), we see that the antibunching intensity is influenced by the interferometer. Moreover, it can be expected that  $S$  will vary if the detector configuration changes, making it possible to increase  $S$  by a smart detector placement. Indeed, it is so as we will show in the following.

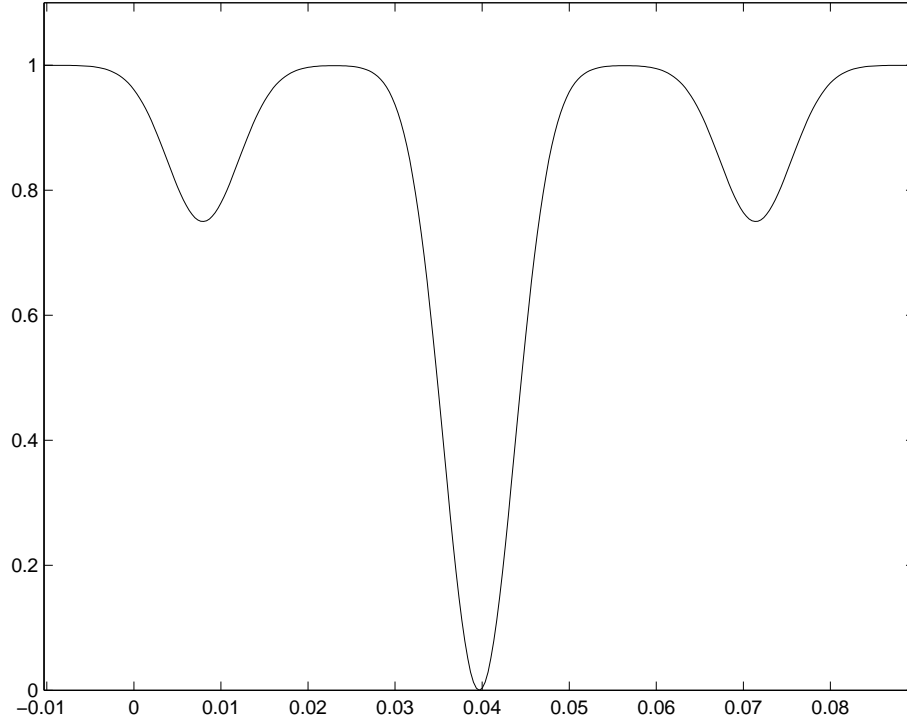


Figure 10.2: The normalized two-electron correlation function as a function of the time difference  $\tau$  expressed in picoseconds for  $\Delta\phi_{1112}(p) - \Delta\phi_{2122}(p) = 2\pi$  and a strongly excited Wien filter. The parameters of the momentum distribution are the same as in Fig. 9.1. The antibunching intensity is three halves of the one in Fig. (9.1).

Let us choose such a detector configuration that the phase differences  $\Delta\phi_{1112}(p)$  and  $\Delta\phi_{2122}(p)$  differ from each other by some integer multiple of  $2\pi$  and vary with  $p$  rapidly even if  $p$  changes over its narrow bandwidth only (later it will become clear how to realize it). Then  $\cos \Delta\phi_{1112}(p) = \cos \Delta\phi_{2122}(p)$  holds and the numerator of the fraction in Eq. (10.19) can be due to the oscillations of the cosine functions evaluated as

$$\int_0^\infty \frac{1}{p} [F(p)]^2 [1 + 2 \cos \Delta\phi_{1112}(p) + \cos^2 \Delta\phi_{1112}(p)] dp \approx \frac{3}{2} \int_0^\infty \frac{1}{p} [F(p)]^2 dp, \quad (10.20)$$

because the integral of the cosine function times  $[F(p)]^2/p$  has canceled and the integral of the square of the cosine function times  $[F(p)]^2/p$  has yielded one half of the integral of  $[F(p)]^2/p$ . The denominator in Eq. (10.19) is simply equal to unity because the cosine terms again cancel by the integration (this causes also vanishing of the interference fringes). The antibunching intensity  $S$  then becomes

$$S \approx \frac{3}{2} \pi \hbar m \int_0^\infty \frac{1}{p} [F(p)]^2 dp = \frac{3}{2} S_0, \quad (10.21)$$

where  $S_0$  would be the antibunching intensity without the biprism (see Eq. (9.38)).

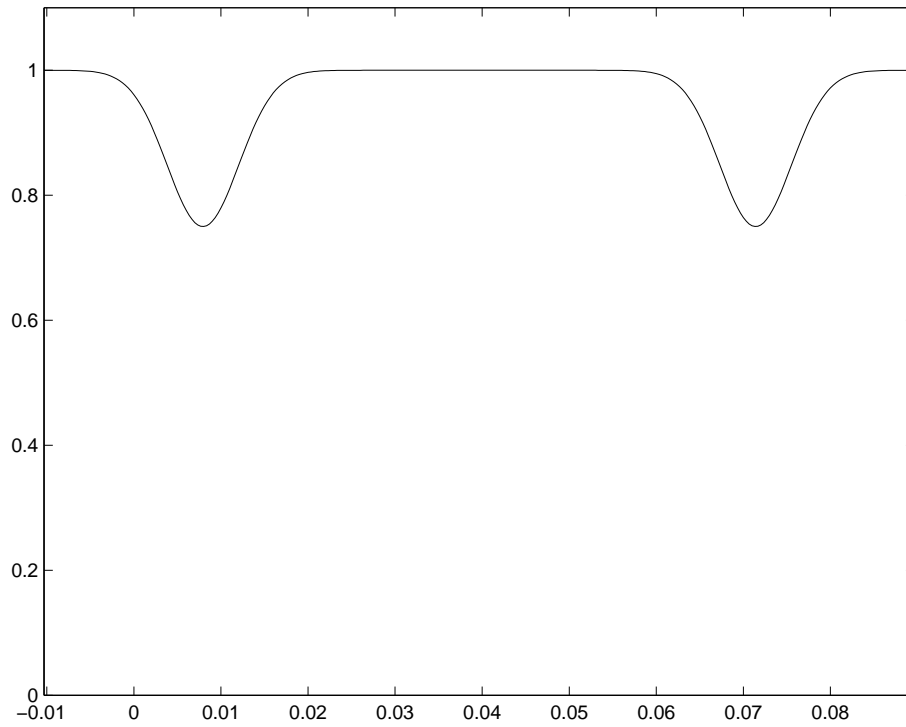


Figure 10.3: The normalized two-electron correlation function as a function of the time difference  $\tau$  expressed in picoseconds for  $\Delta\phi_{1112}(p) - \Delta\phi_{2122}(p) = 3\pi$ . The parameters of the momentum distribution are the same as in Fig. 9.1. The antibunching intensity is one half of the one in Fig. (9.1) and one third of the one in Fig. (10.2).

If we now keep the assumption about the rapid changing of  $\Delta\phi_{1112}(p)$  and  $\Delta\phi_{2122}(p)$  with  $p$  but put  $\Delta\phi_{1112}(p) = \Delta\phi_{2122}(p) + n\pi$ , where  $n$  is an odd number, then  $\cos \Delta\phi_{1112}(p) = -\cos \Delta\phi_{2122}(p)$  and by a similar argumentation we obtain the result

$$S \approx \frac{1}{2} \pi \hbar m \int_0^\infty \frac{1}{p} [F(p)]^2 dp = \frac{1}{2} S_0. \quad (10.22)$$

We see that just by very slight changes of the phases in the experiment it is possible to change the antibunching intensity between one half and three halves of the intensity without the biprism. This shows that the interferometer brings a new quality to electron correlations that could not be achieved by other means. The influence of an interferometer on correlation functions can be even stronger in more complicated interferometers. As has been shown by Silverman [26, 27], in certain cases it is even possible to observe electron bunching.

Now, there is a question how to put the assumptions about the  $\Delta\phi$ 's into practice. This can be achieved by a very useful element in the interferometer, the Wien filter mentioned in Chapter 2. This element is used for introducing longitudinal shifts between the two paths going from the source to a detector. If the Wien filter is strongly excited, the phase differences  $\Delta\phi_{1112}(p)$  and  $\Delta\phi_{2122}(p)$  vary with  $p$  rapidly, which causes vanishing

of the interference fringes<sup>1</sup>. At the same time, by placing the detectors close to each other in the lateral direction and changing their lateral distance, it is possible to change the difference  $\Delta\phi_{1112}(p) - \Delta\phi_{2122}(p)$  very sensitively. For example, if the lateral distance of the detectors is equal to  $s$  times the fringe separation, then  $\Delta\phi_{1112}(p) - \Delta\phi_{2122}(p) = 2\pi s$ . However, we have said there are no more fringes if the Wien filter is strongly excited. This does not really matter because we mean fringes that would exist without the Wien filter. So, putting the detectors to a separation equal to an integer or half-integer multiple of the fringe separation, the antibunching intensity becomes three halves or one half times the intensity without interferometer, respectively. The fact that it is possible to increase the two-electron correlations using the Wien filter is very interesting because in the case of one-electron coherence, the effect of the Wien filter is destructive: the contrast of interference fringes always suffers a decrease under the influence of a Wien filter.

An experiment allowing the verification of our result would not be unthinkable provided that antibunching could be measured at all. We would just have to make sure that each of the two detectors is illuminated by one fringe only and not extended over several fringes. In the latter case the averaging over phases would cancel the effect, as we will see in the following.

The dependence of the normalized two-electron correlation function  $g_{1,2}^{(2)}$  on  $\tau$  is shown in the figures 10.2 and 10.3 for both the discussed cases. It follows from the geometry of the experiment that the Wien filter also moves the whole correlation function along the  $\tau$ -axis. Therefore, even though the detectors are placed at the same distance from the source, the centre of symmetry of the function  $g_{1,2}^{(2)}$  is shifted in time.

### 10.1.2 Average antibunching intensity

Now, there is still a question what happens with the antibunching intensity  $S$  if the detectors are extended over several fringes. Then we have to average  $S$  over the phase differences  $\Delta\phi_{1112}$  and  $\Delta\phi_{2122}$ . However, as the illumination may be different at different places of the interference region, it is necessary to use the weights equal to the product of the one-electron correlation functions for the averaging (we consider now again a general situation without any assumptions about the how much the Wien filter is excited etc.). As factors proportional to these correlation functions stand in the denominator of the fraction (10.19), the weighed averaging of  $S$  is equivalent to a simple averaging of the numerator only. As the phase differences  $\Delta\phi_{1112}(p)$  and  $\Delta\phi_{2122}(p)$  change now independently, all the terms containing the cosine functions give zero. Therefore it holds

$$\langle S \rangle = \pi \hbar m \int_0^\infty \frac{1}{p} [F(p)]^2 dp = S_0. \quad (10.23)$$

We see that the averaged antibunching intensity in the interferometer is equal to the intensity without an interferometer, which could be expected. So, if the detectors are much

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<sup>1</sup>A precise calculation of phase changes of the propagators introduced by the Wien filter and the electron biprism can be found in the diploma thesis of Peter Sonntagag [28].



larger than the fringe separation, no increase or reduction of antibunching intensity should take place.

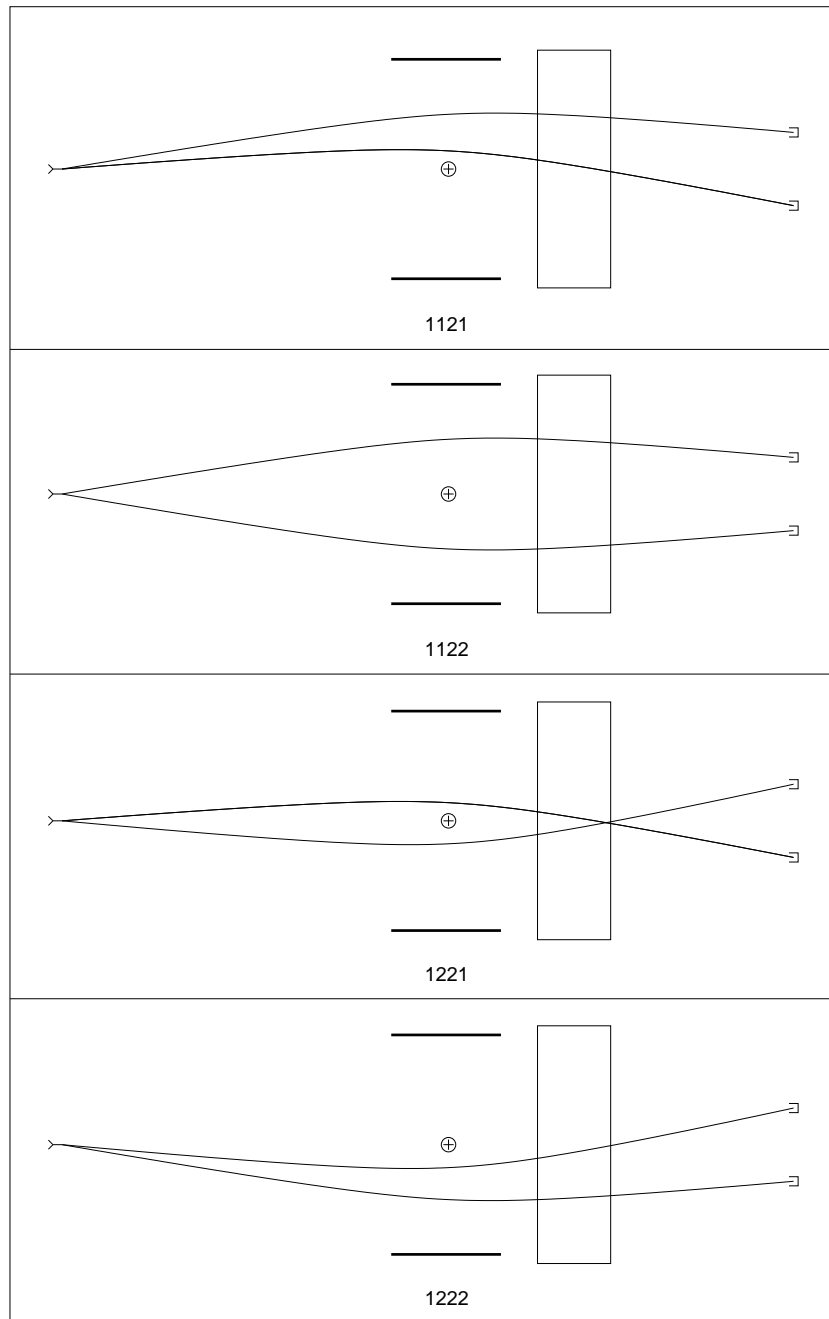


Figure 10.4: The four possibilities how the two electrons can come from the source to the detectors. The corresponding cross-correlation functions are  $\Gamma_{1121}$ ,  $\Gamma_{1122}$ ,  $\Gamma_{1221}$ , and  $\Gamma_{1222}$ , respectively. As we do not distinguish between the four possibilities, all of them interfere and the total cross-correlation function  $\Gamma_{12}$  is equal to the sum of the partial cross-correlation functions.

# Chapter 11

## Can measurement at one detector influence detection at another one?

In this chapter we will discuss an interesting problem that deals with the detection probabilities and counting rates on two electron detectors. We will show that the properties of the two-electron correlation function might seem at first sight to have quite absurd consequences and then try to find the solution of this problem. It should be noted that most considerations in this chapter hold generally and are not limited to quantum-mechanical correlation functions. This is because we deal with quantities such as conditional probabilities that are governed purely by the laws of the classical probability theory. Thus our results can be applied on electrons as well as e.g. projectiles from a machine gun shooting at two “detectors” (which is a good example of classical antibunched particles). The reason why we included such, in some sense “classical” chapter into this dissertation was that it is not unlikely to meet the described problem when one works with correlation functions and in such a case this chapter might be helpful.

### 11.1 Problem

Let  $O_i$  ( $i = 1, 2$ ) denote the time probability density of detecting an electron at the  $i$ th detector at time  $t_i$ . This quantity is at the same time the counting rate at the  $i$ th detector. The probability of detecting an electron at the  $i$ th detector within the time interval  $\langle t_i, t_i + dt_i \rangle$  (we will call such a detecting “event  $i$ ” in the following) is then  $P(i) = O_i dt_i$ . Similarly, let  $O_{12}$  denote the probability density of detecting an electron at *both* the detectors, i.e., at detector 1 at time  $t_1$  and at detector 2 at time  $t_2$ . The probability  $P(2 \wedge 1)$  that we detect one electron at detector 1 within the time interval  $\langle t_1, t_1 + dt_1 \rangle$  and another electron at detector 2 within the time interval  $\langle t_2, t_2 + dt_2 \rangle$  is then equal to  $O_{12} dt_1 dt_2$ . From the fact that  $G_{12}^{(2)} = G_1^{(1)} G_2^{(1)} (1 - |\gamma_{12}|^2)$  according to Eq. (6.30) it follows

$$O_{12} = O_1 O_2 (1 - |\gamma_{12}|^2) \quad (11.1)$$

(see end of section 3.2). Now, suppose that event 1 has occurred, i.e., that we have detected an electron at detector 1 at time  $t_1$ . The probability of detecting another electron

at detector 2 is then equal to the conditional probability  $P(2|1)$  which is according to the well-known formula from the probability theory equal to

$$P(2|1) = \frac{P(2 \wedge 1)}{P(1)} = \frac{O_{12} dt_1 dt_2}{O_1 dt_1} = (1 - |\gamma_{12}|^2) O_2 dt_2. \quad (11.2)$$

The average number of electrons  $N_2(T)$  arriving into detector 2 within a time interval  $\langle t, t+T \rangle$  can be calculated as a time integral of the probability  $P(2|1)$ . For a stationary field,  $O_2$  is a constant in time and therefore

$$N_2(T) = \int_t^{t+T} P(2|1) dt_2 = \int_t^{t+T} (1 - |\gamma_{12}|^2) O_2 dt_2 = O_2 T - O_2 \int_t^{t+T} |\gamma_{12}|^2 dt_2 \quad (11.3)$$

Consider now a simple coincidence experiment without a biprism as described in Chapter 9 and suppose that both the detectors lie within the coherently illuminated area at the same distance from the source. Then  $|\gamma_{12}|^2$  reaches its maximum for  $t_1 = t_2$  and its integral over all  $t_2$  is approximately equal to the coherence time  $T_c$  (let us suppose in the following that it is precisely equal for simplicity). This holds with a good accuracy also for the last integral in Eq. (11.3) if  $t_1 \in \langle t + T_c, t + T - T_c \rangle$ , that is, if the time  $t_1$  is “well” contained in the time interval  $\langle t, t+T \rangle$  because  $|\gamma_{12}|^2$  is practically equal to zero for  $t_2 \notin \langle t_1 - T_c, t_1 + T_c \rangle$ . The average number of counts on detector 2 during the time  $T$  thus becomes  $N_2(T) = O_2(T - T_c)$ , which is less than  $O_2 T$ , however. So, once an electron was detected at the first detector, the number of electrons arriving into the second detector is reduced by  $O_2 T_c$  with respect to the situation when we do not perform any measurement on the first detector. Now, in average there are  $N_1 = O_1 T$  electrons coming to detector 1 during the time interval  $\langle t, t+T \rangle$ . So let us say that electrons have been detected at detector 1 at the times  $t_{11}, t_{12}, \dots, t_{1N_1}$ . Each of these electrons has a similar effect as described above on the number of counts at detector 2<sup>1</sup>. The reason is that the probability density of detecting an electron at detector 2 is reduced for all times  $t_2$  that are close to some moment when an electron at detector 1 was detected. The average reduction of number of counts registered at detector 2 thus becomes approximately  $N_1 O_2 T_c = O_1 O_2 T T_c$  and the relative reduction is  $O_1 O_2 T T_c / (O_2 T) = O_1 T_c$ . It looks very strange: just by turning on one detector (which results in detecting in average  $O_1 T$  electrons within the time  $T$ ), the average number of counts registered at the other detector is reduced by the factor  $O_1 T_c$ , which can be a measurable value. This evidently cannot be true. Indeed, it is not true and we will show why in the following.

## 11.2 Solution

The problem is that when we say that  $N_1 = O_1 T$  electrons were detected at detector 1 at times  $t_{11}, t_{12}, \dots, t_{1N_1}$ , in fact we do not say anything about what happened in between,

<sup>1</sup>in fact, this is correct only if the moments of detections at detector 1 are far from each other compared to the coherence time  $T_c$ , i.e., if  $O_1 T_c \ll 1$ ; in the opposite case we would have to take into account also three-, four-, ... electron correlations

i.e., if maybe some other electrons were also detected. The situation is different when we say that  $N_1$  electrons were detected at detector 1 at times  $t_{11}, t_{12}, \dots, t_{1N_1}$  but *no other* electrons were detected in between. We have seen that the detection on detector 1 *reduces* the counting rate on detector 2. As the counting rate on detector 2 should be unaffected by a measurement on detector 1, we can therefore expect that *no* detection at detector 1 will *enlarge* the counting rate at detector 2. Let us see now if this really happens.

So, let us suppose that *no* electron was detected by detector 1 during the time interval  $\langle t_1, t_1 + dt_1 \rangle$  and let us calculate the probability of detecting an electron at detector 2 with this condition. To do this, we use the fact that the probability of the event 2 (detecting an electron at detector 2 within the time interval  $\langle t_2, t_2 + dt_2 \rangle$ ) can be decomposed into two parts: the probability  $P(2 \wedge 1)$  that event 2 occurs while event 1 (detecting an electron at detector 1 within the time interval  $\langle t_1, t_1 + dt_1 \rangle$ ) also occurs and the probability  $P(2 \wedge 1')$  that event 2 occurs while event 1 does not occur. Mathematically expressed,

$$P(2) = P(2 \wedge 1) + P(2 \wedge 1') \quad (11.4)$$

The conditional probability  $P(2|1')$  that event 2 occurs when event 1 does not occur can be then evaluated using Eq. (11.4) as follows:

$$P(2|1') = \frac{P(2 \wedge 1')}{P(1')} = \frac{P(2) - P(2 \wedge 1)}{1 - P(1)} = \frac{O_2 dt_2 - O_{12} dt_1 dt_2}{1 - O_1 dt_1} = \frac{1 - \frac{O_{12}}{O_2} dt_1}{1 - O_1 dt_1} O_2 dt_2. \quad (11.5)$$

Expanding the denominator of the last fraction and neglecting all the terms of order  $dt^2$  and higher, we get

$$P(2|1') = \left[ 1 + \left( O_1 - \frac{O_{12}}{O_2} \right) dt_1 \right] O_2 dt_2. \quad (11.6)$$

Substituting Eq. (11.1) this into Eq. (11.6), we get

$$P(2|1') = \left[ 1 + \left( O_1 - \frac{O_1 O_2 (1 - |\gamma_{12}|^2)}{O_2} \right) dt_1 \right] O_2 dt_2 = \left( 1 + |\gamma_{12}|^2 O_1 dt_1 \right) O_2 dt_2. \quad (11.7)$$

Eq. (11.7) shows that if *no electron has arrived* to the first detector during the time interval  $\langle t_1, t_1 + dt_1 \rangle$ , the probability that an electron arrives to the second detector *increases*. The fact that this increase is proportional to  $dt_1$  is quite reasonable. The longer is the time interval  $dt_1$ , the larger amount of information we have by knowing that no electron was detected within this interval. Indeed, the term  $O_1 dt_1$  expresses the average number of electrons coming to detector 1 within the time interval  $dt_1$ . Thus, loosely speaking, if we know that *no* electron was detected during the time  $dt_1$ , during which we would normally expect  $O_1 dt_1$  detections, the weight of our information is in some sense equivalent to the weight of information about a detection of one electron multiplied by  $O_1 dt_1$ . Comparing Eqs. (11.7) and (11.2), we then see that the increase of the number of counts on detector 2 in case of *no detection* on detector 1 is the same as the decrease of the number of counts in

the case of a *detection* on detector 1 (both per the same amount of information). Thus, it really happens what we anticipated and we are more ready now to believe that the average counting rate on detector 2 is really unaffected by any measurements performed on detector 1. It should be noted that the presented explanation of the problem just shows where there is a mistake in the argumentation leading to the senseless result. To show precisely that the counting rate on detector 2 is not influenced by any measurement on detector 1 in the outlined way would require a more precise approach using electron correlations of all orders.

# Chapter 12

## Conclusion

### 12.1 Summary

The main aim of this dissertation was to describe the coherence and correlational properties of free electrons in relation to experiment. For this purpose the correlation functions were introduced that express the joint probability of detecting electrons at different places and times. A detail derivation of the correlation functions was given for three states, the chaotic, generalized chaotic and pure ones. Even though the result for the chaotic state is not original, the detail calculation step by step is instructive and can be helpful for calculations of correlation functions for other states. I have not found such a detail calculation in literature and therefore have included it into the dissertation.

As in most published works only spin-polarized or completely unpolarized particles are considered, which is not always the best approximation, the evaluation of the correlation functions with taking the electron spin into account was included. We have seen that the correlation function of order  $k$  for partially polarized electrons can be expressed using the correlation functions of all orders up to  $k$  for polarized electrons and the degree of spin polarization. I assume that this result has not been published yet.

Important inequalities between correlation functions of different orders were then derived using the methods of linear algebra. These inequalities show the fermionic nature of electrons in a very universal and compact way. Namely, they say that the probability of detecting electrons at given places and times can never exceed the probability of detecting electrons at only some of these places and times multiplied with the probability of detecting electrons at the remaining places and times. Either in this case I am not aware that similar inequalities would have been published.

The discussion about the correlation function of the field-emission electron field included results that are mostly generally known. The fact that the angular, longitudinal and temporal extents of the two-electron correlations are equal to the coherence angle, length and time, respectively, was pointed out.

The properties of the two-electron correlation function in an electron interferometer was also discussed. It was shown that the electron correlations can be made stronger or weaker at certain spatial domains with the help of the Wien filter. The question whether

this fact could be used for making the measurement of electron correlations easier remains open. I suppose that the results from this chapter belong among the original ones.

For a little bit recreation a problem with the counting rate was introduced that seemed to lead to a nonsense. It was shown where a mistake in the argumentation was and how to treat the problem properly.

This dissertation does not try by any means to provide a complete theory of electron correlations. I concentrated primarily on correlation functions and omitted e.g. the problems of electron counting statistics. I should also mention that it is possible to build the theory of electron correlations in a different way using the Liouville space formalism [29, 30].

## 12.2 Further research

At the very end of this dissertation, I would like to outline several direction of possible further research.

We have seen that in the theory of electron correlations there are many problems that are still unclear. The most fundamental two of them are the problems of electron emission and detection processes. The problem of field-emission would deserve a very detail analysis covering investigations of the electron state in the metal, the tunneling process, the interaction between the electrons during this process, the influence of the strong electrostatic field in the neighbourhood of the tip on the electron states and many other physical phenomena. I believe that such an analysis is possible even if it will be definitely very complicated.

The problem of the electron detection is of a similar difficulty. Its solution would require especially a precise analysis of the interaction between the detected electron and the material of the detector from which the secondary electrons are emitted. The indistinguishableness between the detected electron and the emitted ones would also have to be taken into account. Of course, the general problem of measurement in quantum mechanics — the reduction of state — plays an important role also in the electron detection process. We can ask, for example, when does exactly this reduction take place? Is it already when the primary electron hits the sensitive layer of the detector, or later, when maybe a coherent superposition of many avalanches of secondary electrons at different places of the microchannel plate has come into existence?

The question of the back-action of the detection process on the electron field is also important. It can, in some sense, enable a better understanding of the electron correlations. Here results from quantum optics (see [31], pp. 69–72) could be probably used successfully.

Last, but not least, also other types of fermions than electrons would deserve our attention concerning their correlational properties. Maybe it will become possible to measure correlations of neutrons in the future, or even of such rare particles as the positrons.



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