Faculty of Science
Masaryk University

Thesis

The Representation of the Variational Sequence by Forms

Michael Krbek

Address:
Department of Theoretical Physics and Astrophysics
Masaryk University
Brno
Czech Republic

E-mail: krbek@physics.muni.cz
WWW: http://www.physics.muni.cz/~krbek

Advisor: Doc. RNDr. Jana Musilová, CSc.

Brno, 2002

Preface

The purpose of this work is to describe the variational sequence formulated on finite order jet prolongations of fibered manifolds. We find the representation of the variational sequence by differential forms generally defined on some higher order jet prolongation of the fibered manifold in question. The findings can be applied to global problems in the calculus of variations such as the inverse problem and variational triviality problem.

I would like to thank my advisor prof. Jana Musilová for her help and prof. Demeter Krupka for fruitful consultations. I would also like to acknowledge the continuing support of the Czech Grant Agency (grant Nr. 201/00/0724) and the Ministry of Education, Youth and Sports (grant Nr. MSM 143100006).

Brno, September 2002

Michael Krbek

Contents

Conventions

Indices and multiindices. The latin indices i, j and their versions with subscripts run through and are summed over from 1 to n unless explicitly specified otherwise. The greek indices σ, ν and their versions with subscripts run through and are summed over from 1 to m unless explicitly specified otherwise.

We use multiindices $I = j_1 \dots j_p$, we denote by $|I|$ the integer p, and call it the length of the multiindex I. We admit multiindices of length zero. For multiindices, we reserve the capital letters I, J and their versions with subscripts.

We use Einstein summation convention, where possible or unless otherwise specified. In order to avoid as much combinatorics in the formulas as possible we shall use the summation over multiindices which is taken over all possible values of the indices. If, for some reason, we want to restrict the summations only to nondecreasingly ordered multiindices, we simply multiply each term in the summation over the multiindex $J = j_1 \dots j_k$ by the number of different ordered k-tuples arising by permuting the set $j_1, \dots, j_k$, i.e.

$$\frac{k!}{p_1! \dots p_n!},$$

where the integers p_i denote the number of integers i, contained in the k-tuple $j_1, \dots, j_k$. The reverse transition from nondecreasingly ordered multiindices to arbitrary multiindices is, of course, also possible.

When we want to indicate that the indices $j_1 \dots j_q$ are symmetrized, antisymmetrized or nondecreasingly ordered, we write $(j_1 \dots j_q)$, $[j_1 \dots j_q]$ and $\{j_1 \dots j_q\}$ respectively.

Geometric structures. We consider manifolds of class C^∞ (smooth). The basic geometric structure used is the fibered manifold, denoted by $\pi\colon Y \to X$ and its r-jet prolongation, denoted by J^rY. By definition, we set $J^0Y = Y$. The natural jet projections will be denoted $\pi^r\colon J^rY \to X$, $\pi^{r,s}\colon J^rY \to J^sY$ for $r > s$. The dimension of the manifold X will be n, the dimension of the manifold Y will be $(m+n)$.

A (local) section of the fibered manifold $\pi\colon Y \to X$, denoted by γ can be prolonged and its r-jet prolongation is denoted by $J^r\gamma$. If $\alpha\colon Y \to Y'$ is a projectable mapping with projection $\alpha_0\colon X \to X'$ of the fibered manifold $\pi\colon Y \to X$ to the fibered manifold $\pi'\colon Y' \to X'$, its prolongation is denoted by $J^r\alpha\colon J^rY \to J^rY'$.

The tangent bundle to a manifold X will be denoted by TX. Let Z be another manifold and $f\colon X \to Z$ a smooth mapping. The tangent mapping will be denoted by $Tf\colon TX \to TZ$. The cotangent bundle to a manifold X will be denoted by T^*X. The qth exterior power of the bundle T^*X will be denoted by $\Lambda_q T^*X$. Le $\pi\colon Y \to X$ be a fibered manifold, $W \subset Y$ an open set and let $W^r = (\pi^{r,0})^{-1}W$. We denote by $\Omega_0^r W$ the ring of functions on W^r. We further denote by $\Omega_q^r W$ the Ω_0^r-module of local sections $W^r \to \Lambda_q T^* J^r W$ of the vector bundle $\Lambda_q T^* J^r Y$.

Geometric operations. We shall use the following notations for standard differential geometric operations.

- The contraction of a differential form ρ by a vector field ξ shall be denoted by $\xi \lrcorner \rho$.
- The exterior product of a differential form ρ and a differential form η shall be denoted by $\rho \wedge \eta$.
- The exterior derivative of a differential form ρ shall be denoted by $\mathrm{d}\,\rho$.
- The Lie derivative of a differential form ρ with respect to the vector field ξ shall be denoted by $\mathscr{L}_\xi \rho$.

Local expressions. When considering local expressions on fibered manifolds, we shall always use charts adapted to the fibered structure $\pi\colon Y \to X$. Such a fibered chart will be denoted by (V, ψ), $\psi = (x^i, y^\sigma)$. On the r-jet prolongation of the fibered manifold Y, we always refer to the associated fibered chart (V^r, ψ^r), where $\psi^r = (x^i, y^\sigma_J)$ for $|J| \leq r$. Such a fibered chart can be canonically constructed from a given fibered chart on Y. When a second fibered chart is needed, we shall denote it by $(\bar{V}^r, \bar{\psi}^r)$, where $\bar{\psi}^r = (\bar{x}^i, \bar{y}^\sigma_J)$ for $|J| \leq r$. We shall say that two such charts overlap if $V \cap \bar{V} \neq \emptyset$.

Components of vector fields and differential forms in bases. A vector field ξ on J^rY shall be written in components

$$\xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma_J \frac{\partial}{\partial y^\sigma_J}.$$

We shall use the symbol

$$\partial^J_\sigma = \frac{\partial}{\partial y^\sigma_{(J)}}$$

for the symmetrized basis vector fields. For a generic differential q-form $\rho \in \Omega^r_q W$ defined on an open set $W^r = (\pi^{r,0})^{-1} W$, where $W \subset Y$ is some open set, we shall write, locally

$$\rho = \sum_{s=0}^{q} P^{J_1 \ldots J_s}_{\sigma_1 \ldots \sigma_s\, i_{s+1} \ldots i_q} \, \mathrm{d}\, y^{\sigma_1}_{J_1} \wedge \ldots \wedge \mathrm{d}\, y^{\sigma_s}_{J_s} \wedge \mathrm{d}\, x^{i_{s+1}} \wedge \ldots \wedge \mathrm{d}\, x^{i_q}.$$

The coefficients $P^{J_1 \ldots J_s}_{\sigma_1 \ldots \sigma_s\, i_{s+1} \ldots i_q} \in \Omega^r_0 W$ are smooth functions on the r-jet prolongation of the fibered manifold Y. They have the obvious symmetry properties

- antisymmetry in the multiindices of the type ${}^{J_p}_{\sigma_p}$,
- antisymmetry in the indices i_l.

Chapter I

Introduction

We follow some developments in the calculus of variations in recent years and try to capture the ideas that have arisen and are relevant to our considerations. Then we focus our attention on the differences between the variational sequence and the variational bicomplex. Finally, we discuss how the present text is sectioned and what the reader can expect in each chapter.

The recapitulation of results. It is of great importance, both for physicists and for mathematicians, to study the Euler–Lagrange formalism in the calculus of variations. An appropriate mathematical setting for the precise formulation appears to be the calculus of real-valued differential forms on jet prolongations of a fibered manifold $\pi\colon Y \to X$. The Lagrangians are usually considered as horizontal forms of degree equal to the dimension $n = \dim X$ of the base manifold X, the equations of motion as differential forms of a degree higher which are one-contact. Such forms are called dynamical. We are especially interested in problems concerning the characterization of the kernel and image of the Euler–Lagrange mapping, i.e. the mapping that maps Lagrangians to their corresponding equations of motion.

The key observation, first to appear in the works of Lepage and Dedecker, relates the Euler–Lagrange mapping to the exterior derivative of differential forms. This idea led Krupka to the concept of Lepage forms in field theory [**23**, **26**] and later to the variational sequence [**34**, **37**, **36**]. This sequence is constructed as the quotient sequence of the well-known de Rham sequence of differential forms which is induced by exterior differentiation and defined on a fixed and finite jet prolongation of a fibered manifold. The base of the fibered manifold can be one-dimensional, this case covers mechanics, or more-dimensional, then we talk about field theory.

The quotient sequence is constructed by means of contact forms (for the degree of the form less or equal to the dimension of the base manifold) and strongly contact forms (for the degree of the form greater than the dimension of the base manifold). The representation of the variational sequence is a mapping whose kernel is exactly the before mentioned quotient sequence. By specifying such a mapping, we obtain the variational sequence from the de Rham sequence by means of a factorization with respect to this mapping. This gives an equivalent description of the variational sequence, moreover, there are some practical advantages to the latter approach, since the objects involved are differential forms, not classes. It is also more in spirit with the applications of the theory.

The representation mapping was constructed by generalizing some techniques taken from the theory of variational bicomplexes, namely the theory of formal (total) differential operators as presented in [**3**, **4**]. There were some previous results in this direction using coordinate calculations. The case of higher order mechanics is considered in [**62**]. The representatives in terms of differential forms for the terms of the variational sequence relevant to the calculus of variations, i.e. the Lagrangian, the dynamical form and their predecessor and successor in the sequence, were given in [**50**] for higher order mechanics and in [**43**] for field theory. The problem of local reconstruction of the classes in higher order mechanics, i.e. the inverse mapping to the representation, was addressed in [**20**] and published in [**51**]. The fact that the local expressions given by [**50**] and [**43**] indeed assemble into well defined differential forms was proved in [**21**].

The global results concerning the variational sequence are most conveniently formulated in the language of sheaves, we consider germs of differential forms on the (fibered) manifold Y. The obstructions allowing to go from local properties of the variational sequence to global properties are given by the de Rham cohomology groups of the manifold Y with coefficients in the constant sheaf $\mathbb{R}$. The local results specify the classes as special polynomials in higher jet variables.

The variational sequence and the variational bicomplex. There exists a slightly different approach to the problem at hand using the variational bicomplex [**3**, **4**, **7**, **45**, **57**, **64**, **65**, **66**]. The variational bicomplex is defined on the infinite jet prolongation of the fibered manifold $\pi\colon Y \to X$, denoted by J^∞. The set $J^\infty Y$ is usually considered with the structure of an infinite dimensional Fréchet manifold, i.e. an infinite dimensional manifold modelled on a locally convex complete metrizable topological vector space E. In this case, E is the product space of symmetric p-linear mappings $\mathrm{Sym}_p(\mathbb{R}^n, \mathbb{R}^m)$ of $\mathbb{R}^n$ into $\mathbb{R}^m$, for all integers $p \geq 0$, considered with the topology of projective limit of the system $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathrm{Sym}_1(\mathbb{R}^n, \mathbb{R}^m)$, $\mathrm{Sym}_2(\mathbb{R}^n, \mathbb{R}^m)$, $\ldots$ with respect to the natural projections. The module of differential q-forms on $J^\infty Y$, constructed by means of the inverse limit with respect to the jet projections $\pi^{\infty,r}\colon J^\infty Y \to J^r Y$ and denoted by Δ_q, can be canonically decomposed into the direct sum of submodules of contact and horizontal forms (compare with (III.3)) denoted by $\Delta^{a,b}$, $a+b=q$. The exterior derivative d admits a decomposition into its horizontal and vertical components accordingly (compare with (III.25,III.26)). The double complex $(\Delta^{a,b}, \mathrm{d_H}, \mathrm{d_C})$ is called the *variational bicomplex*. We can construct the commutative diagram in Figure 1 with locally exact rows and columns.

$$
\begin{tikzcd}
 & & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
 & 0 \arrow[r] & \Delta^{0,2} \arrow[u, "\mathrm{d_C}"] \arrow[r, "\mathrm{d_H}"] & \Delta^{1,2} \arrow[u, "\mathrm{d_C}"] \arrow[r, "\mathrm{d_H}"] & \cdots \arrow[r, "\mathrm{d_H}"] & \Delta^{n-1,2} \arrow[u, "\mathrm{d_C}"] \arrow[r, "\mathrm{d_H}"] & \Delta^{n,2} \arrow[u, "\mathrm{d_C}"] \arrow[r] & \mathscr{E}_2 \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[r] & \Delta^{0,1} \arrow[u, "\mathrm{d_C}"] \arrow[r, "-\mathrm{d_H}"] & \Delta^{1,1} \arrow[u, "\mathrm{d_C}"] \arrow[r, "-\mathrm{d_H}"] & \cdots \arrow[r, "-\mathrm{d_H}"] & \Delta^{n-1,1} \arrow[u, "\mathrm{d_C}"] \arrow[r, "-\mathrm{d_H}"] & \Delta^{n,1} \arrow[u, "\mathrm{d_C}"] \arrow[r] & \mathscr{E}_1 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \Delta^{0,0} \arrow[u, "\mathrm{d_C}"] \arrow[r, "\mathrm{d_H}"] & \Delta^{1,0} \arrow[u, "\mathrm{d_C}"] \arrow[r] & \cdots \arrow[r, "\mathrm{d_H}"] & \Delta^{n-1,0} \arrow[u, "\mathrm{d_C}"] \arrow[r, "\mathrm{d_H}"] & \Delta^{n,0} \arrow[u, "\mathrm{d_C}"] \arrow[r] & \mathscr{E}_0 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \Lambda^0 TX \arrow[u, "(\pi^\infty)^*"] \arrow[r, "\mathrm{d}"] & \Lambda^1 TX \arrow[u, "(\pi^\infty)^*"] \arrow[r, "\mathrm{d}"] & \cdots \arrow[r, "\mathrm{d}"] & \Lambda^{n-1} TX \arrow[u, "(\pi^\infty)^*"] \arrow[r, "\mathrm{d}"] & \Lambda^n TX \arrow[u, "(\pi^\infty)^*"] \arrow[r, "\mathrm{d}"] & 0 \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] & & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}
$$

Figure 1. The variational bicomplex

In every other row, the horizontal component $\mathrm{d_H}$ is taken with the minus sign in order to ensure commutativity from the condition $\mathrm{d_H}\,\mathrm{d_C} = -\,\mathrm{d_C}\,\mathrm{d_H}$ of Proposition (III.8). The Euler–Lagrange mapping in the context of the variational bicomplex is the mapping $\mathscr{E}_0 \to \mathscr{E}_1$, the Helmholtz–Sonin mapping the mapping $\mathscr{E}_1 \to \mathscr{E}_2$. It is also possible to consider finite order variational bicomplexes using pullbacks by $\pi^{r+2,r}$ and Proposition III.8. The general setting for finite order variational bicomplexes is discussed in [67].

The theory on finite order jet spaces keeps the order of operations fixed, except when one considers the representation mapping. Thus, the raising of order by means of the Euler–Lagrange mapping is interpreted as the representation of classes of differential forms on J^rY by differential forms defined on some higher jet prolongation J^sY, $s > r$. This uncovers the analytic structure of the various mappings. The representatives of the classes have also a specific algebraic structure. They are polynomial in fiber variables y^σ_J, $|J| > r$ as can be seen from the formula (V.29) and formulas (V.30,V.31) or the general formula (IV.18).

We shall conclude this paragraph with some remarks concerning the remaining open problems. An important point to make concerns the possibility of generalization of the variational sequence. In principle, one could take another subsequence of the de Rham sequence of differential forms, and by forming a quotient sequence, obtain some different sequence.

There remains an important open problem, namely, the problem of (locally) reconstructing the class from a given representative. The problem was solved in the case of mechanics (see [**20, 51**]). The explicit calculations for field theory leading to such a result are exceedingly tedious, but the main idea is as follows. For the horizontalization h of classes of n-forms the problem was explicitly solved in [**36**]. We assume that the class $\Omega^r_q W/\Theta^r_q W$ is given in some fibered chart by a representative which doesn't contain anything from the quotient $\Theta^r_q W$, i.e.

$$\rho = \sum_{s=0}^{q} P^{K_1\ldots K_s}_{\sigma_1\ldots\sigma_s\; i_{s+1}\ldots i_q}\, \mathrm{d}\, y^{\sigma_1}_{K_1} \wedge \ldots \wedge \mathrm{d}\, y^{\sigma_s}_{K_s} \wedge \mathrm{d}\, x^{i_{s+1}} \wedge \ldots \wedge \mathrm{d}\, x^{i_q},$$

where $|K_s| = r$ and moreover, the systems $P^{K_1\ldots K_s}_{\sigma_1\ldots\sigma_s\; i_{s+1}\ldots i_q}$ are traceless with respect to all possible combinations of latin indices, in order to take into account the formula (III.17). We apply h (resp. I) to the previous expression and obtain polynomials in the jet variables y^σ_J, $r < |J| \leq r+1$ (resp. $r < |J| \leq 2r+1$). Clearly, we can obtain the original traceless systems $P^{K_1\ldots K_s}_{\sigma_1\ldots\sigma_s\; i_{s+1}\ldots i_q}$ by means of a differentiation process with respect to these jet variables y^σ_J, $|J| > r$.

Another interesting question appears to be the connection of the variational sequence with Lepage forms and their generalizations. This question is intimately connected with the decomposition (IV.12).

The contents of chapters. Chapter II contains standard material taken mainly from [**40**] with minor changes in notation. It was prepended mainly for relative completeness of the work. This is true also for the beginning of Chapter III. The results are therefore given without proofs since these can be found in [**40**]. Section 3 of Chapter III is mostly original. We consider the splitting of the pullback of the exterior derivative into horizontal and contact components and define the formal derivative of differential forms.

Chapter IV is based on [4] but we consider the theory for finite prolongations of fibered manifolds. Thus, we need to precisely explain all concepts and also have to modify the formulations and proofs of main theorems. The first section in Chapter V is standard, parts of the second section containing the concept of the variational sequence are taken from [34]. The rest of the chapter about the representation of the variational sequence is original and was submitted to [22].

Chapter II

Fibered Manifolds and Their Prolongations

We introduce fibered manifolds and their prolongations. These are the underlying spaces in the geometric description of the calculus of variations. The projection mapping allows us to define the concept of horizontal forms and vertical vector fields. We define the prolongations of projectable vector fields by prolonging their local one-parameter groups. The material of this chapter is taken from [40] up to minor corrections and notational changes.

1. Fibered Manifolds

1.1. Fibered manifolds. The triple (Y, π, X) is called a *fibered manifold* if Y and X are smooth manifolds and the mapping $\pi : Y \to X$ is a surjective submersion. The smooth manifold X is called the *base*, the smooth manifold Y the *total space* and the mapping $\pi\colon Y \to X$ is called the *projection*. When the projection π is clear from the context we shall denote the fibered manifold (Y, π, X) simply by Y.

Since the tangent mapping $T_y\pi\colon T_yY \to T_{\pi(y)}X$ is surjective at each point $y \in Y$ there exist a chart (V, ψ), $\psi = (u^i, y^\sigma)$ at the point $y \in Y$ and a chart (U, φ), $\varphi = (x^i)$, $U = \pi(V)$ at the point $x = \pi(y)$ such that $x^i\pi = u^i$, i.e. adapted to the projection π. By a slight abuse of notation we can identify the coordinates x^i and u^i and call (V, ψ), $\psi = (x^i, y^\sigma)$ a *fibered chart*. The chart (U, φ), $\varphi = (x^i)$ on X is determined uniquely and is said to be *associated* with the fibered chart (V, ψ). When using expressions involving coordinates throughout this work we shall always be using this chart. The mapping $\gamma : U \to Y$, where $U \subset X$ is an open set, is called a

$$\begin{tikzcd} J^rY \arrow[d, "\pi^{r,0}"] \\ Y \arrow[d, "\pi"'] \\ X \arrow[u, bend right, "\gamma"'] \arrow[uu, bend left=60, "J^r\gamma"] \end{tikzcd}$$

Figure 1. Fibered manifolds and their sections

section of the fibered manifold $\pi \colon Y \to X$ if $\pi \circ \gamma = \mathrm{id}_U$. The situation is depicted in the Figure 1. In fibered coordinates the section γ is given by equations of the form $u^i \circ \gamma = x^i$ and $y^\sigma \circ \gamma = y^\sigma(x^i)$.

1.2. Morphisms and compositions of fibered manifolds. Let Y (resp. $\bar{Y}$) be a fibered manifold with base X (resp. $\bar{X}$) and projection π (resp. $\bar{\pi}$). A mapping $\alpha \colon Y \to \bar{Y}$ is called a *fibered morphism* or simply *morphism* of Y into $\bar{Y}$ if there exists a mapping $\alpha_0 \colon X \to \bar{X}$ such that $\bar{\pi} \circ \alpha = \alpha_0 \circ \pi$ as in the Figure 2. If α_0 exists then it is unique and is called the *projection* of the

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, "\pi"'] & \bar{Y} \arrow[d, "\bar{\pi}"] \\ X \arrow[r, "\alpha_0"'] & \bar{X} \end{tikzcd}$$

Figure 2. Fibered morphisms

fibered morphism α. A fibered morphism which is at the same time a diffeomorphism is called a *fibered isomorphism* or simply *isomorphism*. By a *local isomorphism* of the fibered manifold we mean an isomorphism $\alpha \colon W \to Y$, where $W \subset Y$ is open. The local description of a fibered morphism follows

$$\bar{x}^i = \bar{x}^i(x^j), \qquad \bar{y}^\sigma = \bar{y}^\sigma(x^j, y^\nu). \tag{II.1}$$

Let $\pi \colon Z \to X$ and $\tau \colon Y \to Z$ be fibered manifolds. The fibered manifold $\pi \circ \tau \colon Y \to X$ given by composition of the surjective submersions π and τ is called a *composite fibered manifold*.

1.3. Vertical vectors, projectable vector fields, horizontal forms. Let $\pi\colon Y \to X$ be a fibered manifold. Consider in particular the composite fibered manifold $TY \to TX \to X$. The tangent bundle $TY \to Y$ of the fibered manifold Y has the subbundle $VY = \ker T\pi$, which is called the *vertical bundle* of Y, is elements are called *vertical vectors*.

A differential q-form ρ is said to be π-*horizontal* or simply *horizontal* if $\xi \lrcorner \rho = 0$ for every π-vertical vector $\xi \in TY$. Thus, by definition, the *bundle of horizontal* 1-*forms* $V^*Y \to Y$ is the bundle dual to the vertical tangent bundle $VY \to Y$. In fibered coordinates the vector field Ξ

$$\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma} \tag{II.2}$$

is vertical if and only if $\xi^i = 0$. A differential q-form

$$\begin{aligned}\rho = {} & A_{i_1 \dots i_q} \,\mathrm{d}\, x^{i_1} \wedge \dots \wedge \mathrm{d}\, x^{i_q} + \\ & + \sum_{k=1}^{q} B_{\sigma_{k+1} \dots \sigma_q \; i_1 \dots i_k} \,\mathrm{d}\, y^{\sigma_1} \wedge \dots \wedge \mathrm{d}\, y^{\sigma_k} \wedge \mathrm{d}\, x^{i_{k+1}} \wedge \dots \wedge \mathrm{d}\, x^{i_q}\end{aligned} \tag{II.3}$$

is horizontal if and only if $B_{\sigma_{k+1} \dots \sigma_q \; i_1 \dots i_k} = 0$ for all possible collections of indices $\sigma_1, \dots, \sigma_k$ and $i_{k+1}, \dots, i_q$, where $1 \le k \le q$.

A vector field Ξ on Y is said to be π-*projectable*, or simply *projectable*, if there exists a vector field ξ on the base X such that

$$T\pi \cdot \Xi = \xi \circ \pi \tag{II.4}$$

as in Figure 3. If ξ exists, it is unique, and is called the π-*projection* or

$$\begin{array}{ccc} Y & \xrightarrow{\Xi} & TY \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle T\pi} \\ X & \xrightarrow{\xi} & TX \end{array}$$

Figure 3. Projectable vector fields

simply *projection* of Ξ.

If α_t is the local one-parameter group of Ξ, then it is readily seen that Ξ is π-projectable if and only if each point $y \in Y$ has a neighborhood W such that $\alpha_t\colon W \to Y$ is a local isomorphism of Y for sufficiently small parameters t.

In fibered coordinates a π-projectable vector field Ξ on Y is expressed by

$$\Xi = \xi^i(x^j) \frac{\partial}{\partial x^i} + \Xi^\sigma(x^j, y^\nu) \frac{\partial}{\partial y^\sigma}. \tag{II.5}$$

2. Jet prolongations of fibered manifolds

2.1. Jet prolongations of fibered manifolds. Let $y \in Y$ and let $\Gamma_{(x,y)}$ be the set of all smooth sections γ defined at the point $x \in X$, such that $\gamma(x) = y$. The binary relation $\gamma_1 \sim \gamma_2$ if there exists a fibered chart at the point y such that

$$\frac{\partial^k (y^\sigma \gamma_1 \varphi^{-1})(\varphi(x))}{\partial x^{i_1} \dots \partial x^{i_k}} = \frac{\partial^k (y^\sigma \gamma_2 \varphi^{-1})(\varphi(x))}{\partial x^{i_1} \dots \partial x^{i_k}} \tag{II.6}$$

for all $k \in \{0, \dots, r\}$, is an equivalence on the set $\Gamma_{(x,y)}$. The equivalence class containing the section γ is called an *r-jet* with *source* x and *target* y or the *r-jet of γ at x*, and is denoted $J^r_x \gamma$. We denote by $J^r Y$ the set of r-jets of all sections γ with sources in X and targets in Y. The canonical jet projections are mappings $\pi^{r,s}$ (resp. π^r) of $J^r Y$ onto $J^s Y$, where $0 \le s < r$ (resp. of $J^r Y$ onto X) defined by $\pi^{r,s}(J^r_x \gamma) = J^s_x \gamma$ (resp. $\pi^r(J^r_x \gamma) = x$).

Now we shall introduce the *smooth structure* on $J^r Y$ associated with the smooth structure of Y. The *associated fibered chart* (V^r, ψ^r), $\psi = (x^i, y^\sigma, y^\sigma_{j_1}, \dots, y^\sigma_{j_1 \dots j_r})$ on $J^r Y$ is defined by the two conditions

- **(i)** $V^r = (\pi^{r,0})^{-1}(V)$,
- **(ii)** if $J^r_x \gamma \in V^r$, then

$$y^\sigma_{i_1 \dots i_k}(J^r_x \gamma) = \frac{\partial^k (y^\sigma \gamma \varphi^{-1})(\varphi(x))}{\partial x^{i_1} \dots \partial x^{i_k}}, \tag{II.7}$$

where $k \in \{0, \dots, r\}$.

We shall make extensive use of notation involving multiindices of the form $I = (i_1 \dots i_k)$. The integer $k = |I|$ will be called the *length of the multiindex I*.

Let $(\overline{V}, \overline{\psi})$ from this point onward be another fibered chart on Y such that $V \cap \overline{V} \neq 0$, the *transformation formula* in recurrent form is given by applying the chain rule to the formula (II.7)

$$\overline{y}^\sigma_{i_1 \dots i_k} = \frac{\partial^k (\overline{y}^\sigma \psi^{-1} \circ \psi \gamma \varphi^{-1} \circ \varphi \overline{\varphi}^{-1})(\overline{\varphi}(x))}{\partial \overline{x}^{i_1} \dots \partial \overline{x}^{i_k}}, \tag{II.8}$$

From the chain rule it follows that the coordinate transformation defined in (II.8) is smooth. If the fibered charts (V, ψ) form a smooth atlas on Y, then the associated fibered charts (V^r, ψ^r) define a smooth atlas on the set $J^r Y$ which generates the smooth structure on $J^r Y$ thus making $J^r Y$ a smooth manifold of dimension

$$\dim J^r Y = n + m + nm + m\binom{n+1}{2} + m\binom{n+r-1}{r} = n + m\binom{n+r}{n}.$$

The canonical projections $\pi^{r,s} \colon J^r Y \to J^s Y$ and $\pi^r \colon J^r Y \to X$ are smooth and the triples $(J^r Y, \pi^{r,s}, J^s Y)$, where $0 \le s < r$ and $(J^r Y, \pi^r, X)$ are fibered manifolds. For the above defined fibered manifolds, the concepts of projectable and vertical vector fields as well as horizontal forms are defined analogously as in 1.3.

2.2. Horizontalization of differential forms. We introduce an exterior algebra morphism related to the structure of the jet prolongations of fibered manifolds.

Let ρ be a differential q form on $J^r Y$. There exists one and only one π^r-horizontal q-form $\mathrm{h}\,\rho$ on $J^{r+1}Y$ such that

$$J^r\gamma^*\rho = J^{r+1}\gamma^*\,\mathrm{h}\,\rho. \tag{II.9}$$

The existence of $\mathrm{h}\,\rho$ follows from the definition of the pull-back of differential forms: Let $W \subset Y$ be an open set. If $q = 0$ then $\rho = f$ is, by definition, a function on $J^r Y$ and at each point $J^{r+1}_x\gamma \in (\pi^{r+1,0})^{-1}(W)$

$$\mathrm{h}\,f(J^{r+1}_x\gamma) = f(J^r_x\gamma). \tag{II.10}$$

If $1 \le q$, then for any point $J^{r+1}_x\gamma \in (\pi^{r+1,0})^{-1}(W)$ and any q-tuple of tangent vectors $\Xi_1, \dots, \Xi_q \in T_{J^r_x\gamma}J^r Y$

$$\mathrm{h}\,\rho(J^{r+1}_x\gamma)(\Xi_1, \dots, \Xi_q) = \rho(J^r_x\gamma)(T_x J^r\gamma \cdot T\pi^{r+1} \cdot \Xi_1, \dots, T_x J^r\gamma \cdot T\pi^{r+1} \cdot \Xi_q).$$

Uniqueness follows from writing h down in local coordinates.

Let $W \subset Y$ be an open set. We denote by $\Omega^r_0 W$ the ring of functions on $W^r = (\pi^{r,0})^{-1}(W)$. The $\Omega^r_0 W$-module of differential q-forms on W^r will be denoted by $\Omega^r_q W$. The exterior algebra of forms on W^r will be denoted by $\Omega^r W$.

The horizontalization h can be seen as the morphism of exterior algebras locally induced by horizontalizations of functions and their exterior derivatives.

Theorem II.1. *Let $W \subset Y$ be an open set. The horizontalization $\mathrm{h}\colon \Omega^r W \ni \rho \to \mathrm{h}\,\rho \in \Omega^{r+1} W$ is the unique $\mathbb{R}$-linear, exterior product preserving mapping such that for any function $f\colon W^r \to \mathbb{R}$ and any fibered chart (V, ψ)*

$$\mathrm{h}\,f = f \circ \pi^{r+1,r}, \qquad \mathrm{h}(\mathrm{d}\,f) = \mathrm{D}_i\,f\,\mathrm{d}\,x^i, \tag{II.11}$$

where

$$\mathrm{D}_i\,f = \frac{\partial f}{\partial x^i} + \sum_{|J|=0}^{r} \frac{\partial f}{\partial y^\sigma_J} y^\sigma_{Ji}. \tag{II.12}$$

Proof. The proof can be done in local coordinates and can be found in [40]. □

The function (II.12) $\mathrm{D}_i\, f\colon W^{r+1} \to \mathbb{R}$ is called the *formal derivative* with respect to the coordinate x^i. Note that the function $\mathrm{D}_i\, f$ is affine in the fibered coordinates y^σ_J, where $|J| = r + 1$.

2.3. The horizontalization of tangent vectors. A vector bundle morphism is introduced, acting on the tangent spaces to the jet prolongations of fibered manifolds. This vector bundle morphism is induced by the structure of the jet prolongations.

We assign to every tangent vector $\Xi \in T_{J^{r+1}_x\gamma} J^{r+1}Y$ a tangent vector $\mathrm{h}\,\Xi \in T_{J^r_x\gamma} J^r Y$ by the formula

$$\mathrm{h}\,\Xi = T_x J^r\gamma \circ T\pi^{r+1} \cdot \Xi, \tag{II.13}$$

where $\mathrm{h}\,\Xi$ is independent of the choice of γ.

$$\begin{tikzcd}
J^{r+1}Y \arrow[d] \\
J^rY \arrow[d] \\
\vdots \arrow[d] \\
Y \arrow[d] \\
X \arrow[u, bend left, "\gamma"] \arrow[uuuu, bend right, "J^r\gamma"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
TJ^{r+1}Y \arrow[d] \arrow[dddd, bend left, "T\pi^{r+1}"] \\
TJ^rY \arrow[d] \\
\vdots \arrow[d] \\
TY \arrow[d] \\
TX \arrow[uuu, bend left, "TJ^r\gamma"]
\end{tikzcd}$$

Figure 4. The horizontalization of vector fields

The mapping h is a vector bundle morphism over the jet projection $\pi^{r+1,r}$ and is called π^{r+1}*-horizontalization* or simply *horizontalization*. A tangent vector Ξ is π^{r+1}-vertical if and only if $\mathrm{h}\,\Xi = 0$.

Using the complementary construction we define

$$\mathrm{p}\,\Xi = T\pi^{r+1,r} \cdot \Xi - \mathrm{h}\,\Xi \tag{II.14}$$

and call $\mathrm{p}\,\Xi$ the contact component of Ξ. Clearly $\mathrm{p}\,\Xi$ is a π^r-vertical vector and Ξ is $\pi^{r+1,r}$-vertical if and only if $\mathrm{h}\,\Xi = 0$ and $\mathrm{p}\,\Xi = 0$.

We shall find the expressions for $\mathrm{h}\,\Xi$ and $\mathrm{p}\,\Xi$ in fibered coordinates.

$$\Xi = \xi^i \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{r+1} \Xi^\sigma_I \frac{\partial}{\partial y^\sigma_I} \tag{II.15}$$

and

$$T\pi^{r+1} \cdot \Xi = \xi^i \frac{\partial}{\partial x^i}. \tag{II.16}$$

Using the identities

$$\frac{\partial (x^j \circ J^r\gamma)}{\partial x^i} = \delta^j_i \quad \text{and} \quad \frac{\partial y^\sigma_I \circ J^r\gamma}{\partial x^i} = y^\sigma_{Ii} \circ J^{r+1}\gamma \tag{II.17}$$

we obtain

$$\mathrm{h}\,\Xi = \xi^j \left(\frac{\partial}{\partial x^j} + \sum_{|I|=0}^{r} y^\sigma_{Ij} \frac{\partial}{\partial y^\sigma_I} \right). \tag{II.18}$$

We readily observe other properties of the mappings h and p. Let $\Xi \in T_{J^{r+2}_x\gamma} J^{r+2}Y$ be a tangent vector. Then

$$\mathrm{h}\,\mathrm{h}\,\Xi = T_x J^r\gamma \circ T\pi^{r+2} \cdot \Xi \tag{II.19}$$

$$\mathrm{p}\,\mathrm{h}\,\Xi = 0 \tag{II.20}$$

$$\mathrm{h}\,\mathrm{p}\,\Xi = 0 \tag{II.21}$$

$$\mathrm{p}\,\mathrm{p}\,\Xi = T\pi^{r+2,r} \cdot \Xi - \mathrm{h}\,\mathrm{h}\,\Xi. \tag{II.22}$$

Remark 1. We may pose the question whether the operations h and p make sense for vector fields. The answer is affirmative, although we need the concept of a vector field along a map. Let Y, Z be manifolds and $f\colon Y \to Z$ a smooth map. A *vector field along the map* f is defined as the map $\Xi\colon Y \to TZ$ so that the diagram in Figure 5 commutes (p denotes the canonical foot projection).

$$\begin{tikzcd} & TZ \arrow[d, "p"] \\ Y \arrow[ur, "\Xi"] \arrow[r, "f"'] & Z \end{tikzcd}$$

Figure 5. Vector field along a map

Indeed, let Ξ be a vector field on $J^{r+1}Y$. Then $\mathrm{h}\,\Xi$ and $\mathrm{p}\,\Xi$ are vector fields along the natural projection $\pi^{r+1,r}\colon J^{r+1}Y \to J^rY$.

3. Jet prolongations of projectable vector fields

3.1. Jet prolongations of local isomorphisms. Let $W \subset Y$ be an open set. Let $\alpha\colon W \to Y$ be a local isomorphism of Y, $W_0 = \pi(W)$, and

$\alpha_0\colon W_0 \to X$ the projection of α. Let $W^r = (\pi^{r,0})^{-1}(W)$. We define a local isomorphism $J^r\alpha\colon W^r \to J^rY$ of J^rY by

$$J^r\alpha(J^r_x\gamma) = J^r_{\alpha_0(x)}\alpha \circ \gamma \circ \alpha_0^{-1}. \tag{II.23}$$

$J^r\alpha$ is called the *r-jet prolongation* of α. The prolongations of local isomorphisms α and β have the following properties

$$\begin{tikzcd}
W^r \arrow[r, "J^r\alpha"] \arrow[d] & J^rY \arrow[d] \\
W \arrow[r, "\alpha"] & Y \\
W_0 \arrow[u, "\gamma"'] \arrow[uu, bend left, "J^r\gamma"] & X \arrow[l, "\alpha_0^{-1}"'] \arrow[u, "\alpha\circ\gamma\circ\alpha_0^{-1}"'] \arrow[uu, bend right, "J^r\alpha\circ\gamma\circ\alpha_0^{-1}"']
\end{tikzcd}$$

Figure 6. Prolongations of local isomorphisms

$$J^r\alpha \circ J^r\gamma \circ \alpha_0^{-1} = J^r\alpha\gamma\alpha_0^{-1}, \tag{II.24}$$

$$\pi^r \circ J^r\alpha = \alpha_0 \circ \pi^r, \tag{II.25}$$

$$\pi^{r,s} \circ J^r\alpha = J^s\alpha \circ \pi^{r,s}, \tag{II.26}$$

$$J^r\alpha \circ J^r\beta = J^r(\alpha \circ \beta). \tag{II.27}$$

We describe the r-jet prolongation of a local isomorphism α in fibered coordinates. We assume that $\alpha\colon W \to Y$ is expressed in a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, by equations

$$\hat{x}^i = x^i\alpha = f^i(x^j), \qquad \hat{y}^\sigma = y^\sigma\alpha = g^\sigma(x^j, y^\nu). \tag{II.28}$$

The image $J^r\alpha(J^r_x\gamma)$ has the coordinates

$$x^i \circ J^r\alpha(J^r_x\gamma) = x^i \circ (J^r_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}) = x^i \circ \alpha_0(x) = f^i(x^j), \tag{II.29}$$

$$y^\sigma \circ J^r\alpha(J^r_x\gamma) = y^\sigma \circ (J^r_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}) = y^\sigma \circ \alpha(\gamma(x)) = g^\sigma(x^j, y^\nu), \tag{II.30}$$

$$y^\sigma_I \circ J^r\alpha(J^r_x\gamma) = y^\sigma_J \circ (J^r_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}) = \frac{\partial^k y^\sigma(\alpha\gamma\alpha_0^{-1})\varphi^{-1}(\varphi\alpha_0(x))}{\partial\hat{x}^{i_1}\dots\hat{x}^{i_k}} \tag{II.31}$$

$$= \frac{\partial^k \hat{y}^\sigma\gamma\alpha_0^{-1}\varphi^{-1}(\hat{x}^j)}{\partial\hat{x}^{i_1}\dots\hat{x}^{i_k}} \tag{II.32}$$

and we obtain a recurrent formula. If the rules for $l < r$ are already given by

$$y^\sigma_I J^r\alpha = g^\sigma_J(x^j, y^\nu_J), \quad |I| \le l, \quad 0 \le |J| \le |I|, \tag{II.33}$$

then the chart expression of $J^r\alpha$ is given by

$$y^\sigma_{Ij} J^r\alpha = \mathrm{D}_i\, g^\sigma_J(x^j, y^\nu_J)\frac{\partial x^i}{\partial\hat{x}^j}, \quad 0 \le |I| \le r-1, \tag{II.34}$$

3.2. Jet prolongations of projectable vector fields. Let Ξ be a π-projectable vector field on Y, ξ its π-projection, α_t the local one-parameter group of Ξ and $J^r\alpha_t$ the r-jet prolongation of α_t. For every point $J^r_x\gamma \in \operatorname{dom} J^r\alpha_t$ we define

$$J^r\Xi(J^r_x\gamma) = \left[\frac{\mathrm{d}}{\mathrm{d}\,t} J^r\alpha_t(J^r_x\gamma)\right]_{t=0}. \tag{II.35}$$

The vector field $J^r\Xi$ on J^rY is called the *r-jet prolongation* of Ξ. It follows from the definition that $J^r\Xi$ is π^r-projectable (resp. $\pi^{r,s}$-projectable, for $r > s$) and its π^r-projection is ξ (resp. $J^s\Xi$). The description of the local structure of jet prolongations of projectable vector fields is presented in the following lemma.

Lemma II.2. *Let Ξ be a π-projectable vector field on Y and let Ξ be expressed in fibered coordinates by*

$$\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma}. \tag{II.36}$$

Then

$$J^r\Xi = \xi^i \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{r} \Xi^\sigma_I \frac{\partial}{\partial y^\sigma_I}, \tag{II.37}$$

where the components Ξ^σ_I are determined by the recurrent formula

$$\Xi^\sigma_{Ii} = \mathrm{D}_i\, \Xi^\sigma_I - y^\sigma_{Ij} \frac{\partial \xi^j}{\partial x^i}. \tag{II.38}$$

The following Lemma concerns the *Lie bracket* operation on r-jet prolongations of projectable vector fields and the *Lie derivatives* by such vector fields.

Lemma II.3. *Let Ξ and Υ be two π-projectable vector fields on Y, ρ a differential form on J^rY.*

(a) *The Lie bracket $[J^r\Xi, J^r\Upsilon]$ is also π-projectable and*

$$J^r[\Xi, \Upsilon] = [J^r\Xi, J^r\Upsilon]. \tag{II.39}$$

(b) *For the Lie derivative of ρ it holds*

$$\mathscr{L}_{J^{r+1}\Xi}\, \mathrm{h}\,\rho = \mathrm{h}\,\mathscr{L}_{J^r\Xi}\rho. \tag{II.40}$$

Lemma II.4. *Let Ξ and Υ be two π-projectable vector fields on Y such that $\Xi \circ \gamma = \Upsilon \circ \gamma$ for a section γ of Y. Then*

$$J^r\Xi \circ J^r\gamma = J^r\Upsilon \circ J^r\gamma. \tag{II.41}$$

Chapter III

Contact Forms

We develop the canonical decomposition of differential forms on the r-jet prolongations of fibered manifolds. This decomposition allows to decompose the natural pull-back of a differential form to the $(r+1)$-jet prolongation of the fibered manifold into its contact components of different degrees of contactness and to decompose the natural pull-back to the $(r+2)$-jet prolongation of the fibered manifold of the exterior derivative of a differential form into its horizontal and contact components. For our purposes we introduce the concept of formal derivatives of differential forms by means of the afore mentioned decomposition. Otherwise, this chapter is the recapitulation of some results obtained in [40] which are relevant to our considerations.

1. The canonical decomposition of forms

1.1. The canonical decomposition of forms on jet bundles. Let $W \subset Y$ be an open set and let us consider the horizontalization of vectors $\mathrm{h}\colon TJ^{r+1}Y \to TJ^rY$ defined in the previous chapter. The mapping h induces a decomposition of each of the modules of q-forms $\Omega^r_q W$. Let $\rho \in \Omega^r_q W$ be a q-form and let $\Xi_1, \dots, \Xi_q$ be a q-tuple of vectors at a point $J^{r+1}_x\gamma \in W^{r+1}$. For each $l \in \{1, \dots, q\}$ we write

$$T\pi^{r+1,r} \cdot \Xi_l = \mathrm{h}\,\Xi_l + \mathrm{p}\,\Xi_l. \tag{III.1}$$

Since

$$(\pi^{r+1,r})^* \rho(J_x^{r+1}\gamma)(\Xi_1, \dots, \Xi_q) = \rho(J_x^r\gamma)(T\pi^{r+1,r} \cdot \Xi_1, \dots, T\pi^{r+1,r}\Xi_q),$$

we can collect the terms homogeneous of degree $(q-k)$ in the horizontal components $\mathrm{h}\,\Xi_1, \dots, \mathrm{h}\,\Xi_q$ and obtain a q-form on W^{r+1} defined by

(III.2) $$\mathrm{p}_k\, \rho(J_x^{r+1}\gamma)(\Xi_1, \dots, \Xi_q) = \\ = \tfrac{1}{k!(q-k)!}\varepsilon^{l_1 \dots l_k l_{k+1} \dots l_q} \rho(J_x^r\gamma)(\mathrm{p}\,\Xi_{l_1}, \dots, \mathrm{p}\,\Xi_{l_k}, \mathrm{h}\,\Xi_{l_{k+1}}, \dots, \mathrm{h}\,\Xi_{l_q}).$$

The form $\mathrm{p}_k\,\rho$ is called the *k-contact component* of the form ρ. Notice that the mapping $\mathrm{p}_0\,\rho$ coincides with the π^r-horizontalization $\mathrm{h}\,\rho$ defined in (II.9) and that we can extend the definition of h to functions by defining for $f \in \Omega_0^r W$, $\mathrm{h}\, f = f \circ \pi^{r+1,r}$. The following formula for $\rho \in \Omega_q^r W$ is at hand.

(III.3) $$(\pi^{r+1,r})^* \rho = \sum_{k=0}^{q} \mathrm{p}_k\, \rho.$$

This formula will be referred to as the *canonical decomposition* of the form ρ.

The mappings $\mathrm{p}_k \colon \Omega_q^r W \to \Omega_q^{r+1} W$ are linear over the ring of functions but they are *not* morphisms of exterior algebras of forms. One can derive their behavior with respect to the exterior product by means of the formula

(III.4) $$(\pi^{r+1,r})^*(\rho \wedge \eta) = (\pi^{r+1,r})^* \rho \wedge (\pi^{r+1,r})^* \eta.$$

If we denote $\mathrm{p}_k\, \Omega_q^r W$ the image of $\Omega_q^r W$ in $\Omega_q^{r+1} W$, we get the direct sum of submodules

(III.5) $$(\pi^{r+1,r})^* \Omega_q^r W = \mathrm{h}\, \Omega_q^r W \oplus \mathrm{p}_1\, \Omega_q^r W \oplus \dots \oplus \mathrm{p}_q\, \Omega_q^r W.$$

1.2. Contact forms. A q-form is called *contact* if $\mathrm{h}\,\rho = 0$. A function $f \in \Omega_0^r W$ is contact if and only if $f = 0$. The form ρ is called *k-contact* if $(\pi^{r+1,r})^* \rho = \mathrm{p}_k\, \rho$ or equivalently if $\mathrm{p}_l\, \rho = 0$ for every $l \neq k$. The integer k is called the *degree of contactness* of the form ρ. We say that ρ is of the *order of contactness* k if $\mathrm{p}_l\, \rho = 0$ for $0 \leq l \leq k-1$.

Lemma III.1. *Let $W \subset Y$ be an open set. The q-form $\rho \in \Omega_q^r W$ is contact if and only if*

(III.6) $$J^r\gamma^* \rho = 0$$

for every smooth section γ of Y defined on W. The form ρ is π^r-horizontal if and only if $\mathrm{p}_l\, \rho = 0$ for $1 \leq l \leq q$. Each of the forms $\mathrm{p}_l\, \rho$, where $1 \leq l \leq q$, is contact.

If $q \geq 1$ and $\rho \in \Omega_q^r W$ is a contact form, then by (III.6), $\mathrm{d}\,\rho$ is also a contact form, and, for any p-form $\eta \in \Omega_p^r W$, the exterior product $\rho \wedge \eta$ is also a contact form. The contact forms constitute a differential ideal in the exterior algebra $\Omega^r W$. This ideal is called the *contact ideal* and it is denoted by $\operatorname{cont}\Omega^r W$.

Example 1. We discuss the important example of contact 1-forms. We define

$$\omega^\sigma_J = \mathrm{d}\, y^\sigma_J - y^\sigma_{Ji}\, \mathrm{d}\, x^i, \quad \text{where} \quad 0 \le |J| \le r-1. \tag{III.7}$$

One can easily compute that for any tangent vector $\Xi \in T_{J^{r+1}_x\gamma} J^{r+1}Y$

$$\omega^\sigma_J(J^r_x\gamma)(\mathrm{h}\,\Xi) = 0 \tag{III.8}$$

$$\omega^\sigma_J(J^r_x\gamma)(\mathrm{p}\,\Xi) = \Xi^\sigma_J - y^\sigma_{Ji}\xi^i \tag{III.9}$$

and moreover

$$\mathrm{d}\, x^i(J^r_x\gamma)(\mathrm{h}\,\Xi) = \xi^i \tag{III.10}$$

$$\mathrm{d}\, x^i(J^r_x\gamma)(\mathrm{p}\,\Xi) = 0. \tag{III.11}$$

1.3. The local structure of contact components. If $\rho \in \Omega^r_q W$ is a differential q-form on J^rY we usually write it in fibered coordinates on W^r as

$$\rho = \sum_{s=0}^{q} P^{J_1\dots J_s}_{\sigma_1\dots\sigma_s\; i_{s+1}\dots i_q}\, \mathrm{d}\, y^{\sigma_1}_{J_1} \wedge \dots \wedge \mathrm{d}\, y^{\sigma_s}_{J_s} \wedge \mathrm{d}\, x^{i_{s+1}} \wedge \dots \wedge \mathrm{d}\, x^{i_q}. \tag{III.12}$$

The coefficients $P^{J_1\dots J_s}_{\sigma_1\dots\sigma_s\; i_{s+1}\dots i_q}$, $s \in \{1,\dots,q\}$ are functions on J^rY antisymmetric in the multiindices ${}^{J_l}_{\sigma_l}$ and in the indices $i_{s+1},\dots,i_q$, where $l \in \{1,\dots,s\}$.

Theorem III.2. *Let $W \subset Y$ be an open set, $q > 1$ an integer $\rho \in \Omega^r_q W$ a q-form with the chart expression (III.12). Then $\mathrm{p}_k\,\rho$ has the chart expression*

$$\mathrm{p}_k\,\rho = Q^{J_1\dots J_k}_{\sigma_1\dots\sigma_k\; i_{k+1}\dots i_q}\, \omega^{\sigma_1}_{J_1} \wedge \dots \wedge \omega^{\sigma_k}_{J_k} \wedge \mathrm{d}\, x^{i_{k+1}} \wedge \dots \wedge \mathrm{d}\, x^{i_q}, \tag{III.13}$$

where

(III.14)
$$Q^{J_1\dots J_k}_{\sigma_1\dots\sigma_k\; i_{k+1}\dots i_q} = \sum_{s=k}^{q} \binom{s}{k} P^{J_1\dots J_{k+1}\dots J_s}_{\sigma_1\dots\sigma_{k+1}\dots\sigma_s\;[i_{s+1}\dots i_q} y^{\sigma_{k+1}}_{J_{k+1}i_{k+1}} \dots y^{\sigma_s}_{J_s i_s]}$$

are functions on J^rY antisymmetrized in $[i_{k+1}\dots i_q]$.

Remark 2. Note that the expressions on the right hand side of the equation (III.14) are polynomials in the jet variables y^σ_J for $|J| = r+1$.

Remark 3. The mappings $\mathrm{p}_l \colon \Omega^r_q W \to \Omega^{r+1}_q W$ for $0 \le l \le q$ behave almost like projectors, i.e.

$$\mathrm{p}_l\, \mathrm{p}_k\, \rho = \delta_{kl}(\pi^{r+2,r+1})^*\, \mathrm{p}_k\, \rho. \tag{III.15}$$

1.4. Strongly contact forms. Let $n+1 \leq q \leq \dim J^rY$ and $\rho \in \Omega^r_q W$ be a q-form. Then, using (III.3), $(\pi^{r+1,r})^*\rho = \mathrm{p}_{q-n}\,\rho + \dots + \mathrm{p}_q\,\rho$. Consequently, $\rho \in \Omega^r_q W$ is always contact. We say that the form ρ is *strongly contact* if $\mathrm{p}_{q-n}\,\rho = 0$. Strongly contact forms are locally characterized by the property that the coefficient of the terms containing the n-form $\mathrm{d}\,x^{i_1} \wedge \ldots \wedge \mathrm{d}\,x^{i_n}$ (the local volume element on X) is always zero.

Remark 4. The exterior differentiation preserves the property of contactness. This no longer holds for strong contactness. Consider for example $n = 2$ and the 3-form $\omega^\sigma \wedge \mathrm{d}\,\omega^\nu$ on Y. By lifting ρ to J^1Y we obtain $(\pi^{1,0})^*\rho = -\omega^\sigma \wedge \omega^\nu_j \wedge \mathrm{d}\,x^j$ and see that ρ is strongly contact. But $\mathrm{d}\,\rho = \mathrm{d}\,\omega^\sigma \wedge \mathrm{d}\,\omega^\nu$ and $(\pi^{1,0})^*\,\mathrm{d}\,\rho = -\omega^\sigma_i \wedge \omega^\nu_j \wedge \mathrm{d}\,x^i \wedge \mathrm{d}\,x^j$ so $\mathrm{d}\,\rho$ is not strongly contact.

1.5. Contact components and geometric operations. We shall study the geometric operations on the contact components such as the exterior product $\wedge$, the contraction $\lrcorner$ of a form with a vector, The Lie derivative $\mathscr{L}$ by a vector field and the exterior derivative.

Lemma III.3. *Let ρ and η be two differential forms on J^rY, $J^{r+1}_x\gamma \in J^{r+1}Y$ a point, Υ a tangent vector at this point, $\alpha : W^r \to J^rY$ a fibered morphism and Ξ a π-projectable vector field on Y. Then*

(a) $\mathrm{p}_k(\rho \wedge \eta) = \sum_{l+s=k} \mathrm{p}_l\,\rho \wedge \mathrm{p}_s\,\eta$,
(b) $\Upsilon \lrcorner\, \mathrm{p}_k\,\rho(J^{r+1}_x\gamma) = \mathrm{p}_{k-1}\,\mathrm{p}\,\Upsilon \lrcorner\, \rho(J^{r+1}_x\gamma) + \mathrm{p}_k\,\mathrm{h}\,\Upsilon \lrcorner\, \rho(J^{r+1}_x\gamma)$,
(c) $\mathrm{p}_k(J^r\alpha^*\rho) = (J^{r+1}\alpha)^*\,\mathrm{p}_k\,\rho$,
(d) $\mathrm{p}_k(\mathscr{L}_{J^r\Xi}\rho) = \mathscr{L}_{J^{r+1}\Xi}\,\mathrm{p}_k\,\rho$,
(e) $(\pi^{r+2,r+1})^*\,\mathrm{p}_k\,\mathrm{d}\,\rho = \mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_{k-1}\,\rho + \mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_k\,\rho$.

Remark 5. If Υ is a π^{r+1}-vertical, $\pi^{r+1,r}$-projectable vector field with $\pi^{r+1,r}$-projection Υ_0. Then $\mathrm{p}\,\Upsilon = \Upsilon_0$ and

$$\Upsilon \lrcorner\, \mathrm{p}_k\,\rho = \mathrm{p}_{k-1}\,\Upsilon_0 \lrcorner\, \rho. \tag{III.16}$$

2. The local structure of contact and strongly contact forms

2.1. The local structure of contact forms. We characterize the local structure of contact q-forms, where $1 \leq q \leq n$. Special attention is paid to the local generators of the contact ideal. Note that for every multiindex J,

$$\mathrm{d}\,\omega^\sigma_J = -\,\mathrm{d}\,y^\sigma_{Ji} \wedge \mathrm{d}\,x^i = -\omega^\sigma_{Ji} \wedge \mathrm{d}\,x^i. \tag{III.17}$$

Thus the 2-forms $\mathrm{d}\,\omega^\sigma_J$ with $|J| \leq r-2$ belong to the ideal of forms generated by the contact 1-forms (III.7) whereas the 2-forms $\mathrm{d}\,\omega^\sigma_J$, for $|J| = r-1$, do *not* belong to the ideal generated by the contact 1-forms in spite of the fact that they are obviously contact. We recall that for $q > n$ *every* form $\rho \in \Omega^r_q W$ is contact.

Theorem III.4. *Let $W \subset Y$ be an open set, $\rho \in \Omega^r_q W$ a differential q-form.*

(a) Let $q = 1$. Then ρ is contact if and only if

$$\rho = \sum_{|J|=0}^{r-1} A^J_\sigma \omega^\sigma_J \tag{III.18}$$

for some functions $A^J_\sigma \in \Omega^r_0 W$.

(b) Let $2 \leq q \leq n$, Then ρ is contact if and only if

$$\rho = \sum_{|J|=0}^{r-1} \omega^\sigma_J \wedge A^J_\sigma + \mathrm{d}\,\omega^\sigma_I \wedge B^I_\sigma, \qquad \textit{where the length of } |I| = r-1 \tag{III.19}$$

and A^J_σ (resp. B^I_σ) are some local $(q-1)$-forms (resp. $(q-2)$-forms).

2.2. The local structure of strongly contact forms. We characterize the local structure of strongly contact q-forms, where $n+1 \leq q \leq \dim J^r Y$.

Theorem III.5. *Let $W \subset Y$ be an open set, q an integer such that $n+1 \leq q \leq \dim J^r Y$ and $\rho \in \Omega^r_q W$ a differential q-form. Then ρ is strongly contact if and only if*

$$\rho = \sum_{\substack{q-n+1 \leq p+s \\ p+2s \leq q}} \omega^{\sigma_1}_{J_1} \wedge \ldots \wedge \omega^{\sigma_p}_{J_p} \wedge \mathrm{d}\,\omega^{\nu_1}_{I_1} \wedge \ldots \wedge \mathrm{d}\,\omega^{\nu_s}_{I_s} \wedge A^{J_1 \ldots J_p\, I_1 \ldots I_s}_{\sigma_1 \ldots \sigma_p\, \nu_1 \ldots \nu_s}, \tag{III.20}$$

where $|J_1|, \ldots, |J_p| \leq r-1$, $|I_1|, \ldots, |I_s| = r-1$ and $A^{J_1 \ldots J_p\, I_1 \ldots I_s}_{\sigma_1 \ldots \sigma_p\, \nu_1 \ldots \nu_s}$ are some local $(q-p-2s)$-forms.

2.3. Decomposable forms. The canonical decomposition (III.3) of a differential form on $J^r Y$ is generally defined on $J^{r+1} Y$. We can consider forms on $J^r Y$ which admit such decomposition directly on $J^r Y$.

A form $\rho \in \Omega^r_q W$ is said to be *decomposable* if $\rho = \alpha + \beta$, $\alpha \in \Omega^r_q W$ is π^r-horizontal (resp. $(q-n)$-contact) and β is contact (resp. strongly contact) for $1 \leq q \leq n$ (resp. $n+1 \leq q \leq \dim J^r Y$).

Theorem III.6. *Let $W \subset Y$ be an open set.*

(a) Let $1 \leq q \leq n$. A q-form $\rho \in \Omega^r_q W$ is decomposable if and only if there exists a π^r-horizontal form $\eta \in \Omega^r_q W$ such that $\mathrm{h}(\rho - \eta) = 0$.

(b) Let $n+1 \leq q \leq \dim J^r Y$. A q-form $\rho \in \Omega^r_q W$ is decomposable if and only if there exists a $(q-n)$-contact form $\eta \in \Omega^r_q W$ such that $\mathrm{p}_{q-n}(\rho - \eta) = 0$.

2.4. The local structure of decomposable forms. We describe the local structure of decomposable forms on J^rY.

Theorem III.7. *Let $W \subset Y$ be an open set and $\rho \in \Omega^r_q W$ a q-form.*

(a) Let $q = 1$. Then ρ is decomposable if and only if

$$\rho = \alpha + \sum_{|J|=0}^{r-1} A^J_\sigma \omega^\sigma_J, \tag{III.21}$$

α is a local π^r-horizontal q-form and the A^J_σ are local functions.

(b) Let $2 \le q \le n$. Then ρ is decomposable if and only if

$$\rho = \alpha + \sum_{|J|=0}^{r-1} \omega^\sigma_J \wedge A^J_\sigma + \omega^\sigma_I \wedge B^I_\sigma, \quad |I| = r-1, \tag{III.22}$$

α is a local π^r-horizontal q-form and the $A^J_\sigma \in \Omega^r_{q-1}W$ (resp. $B^I_\sigma \in \Omega^r_{q-2}W$ are local $(q-1)$-forms (resp. $(q-2)$-forms).

(c) Let $n+1 \le q \le \dim J^rY$. Then ρ is decomposable if and only if

$$\rho = \alpha + \sum_{\substack{q-n+1 \le p+s \\ p+2s \le q}} \omega^{\sigma_1}_{J_1} \wedge \ldots \wedge \omega^{\sigma_p}_{J_p} \wedge \mathrm{d}\,\omega^{\nu_1}_{I_1} \wedge \ldots \wedge \mathrm{d}\,\omega^{\nu_s}_{I_s} \wedge A^{J_1 \ldots J_p\, I_1 \ldots I_s}_{\sigma_1 \ldots \sigma_p\, \nu_1 \ldots \nu_s}, \tag{III.23}$$

$\alpha \in \Omega^r_q W$ is a local $(q-n)$-contact q-form, $|J_1|, \ldots, |J_p| \le r-1$, $|I_1|, \ldots, |I_s| = r-1$ and $A^{J_1 \ldots J_p\, I_1 \ldots I_s}_{\sigma_1 \ldots \sigma_p\, \nu_1 \ldots \nu_s} \in \Omega^r_{q-p-2s}W$ are some local $(q-p-2s)$-forms.

3. Decomposition of the exterior derivative

We will prove that the operator $(\pi^{r+2,r})^*\,\mathrm{d}$ can be decomposed into its horizontal and contact component. By means of this decomposition, we shall extend the notion of the formal derivative from functions to differential forms. We shall investigate the properties of such derivatives, give some examples and calculate the effect of these derivatives on basis 1-forms.

3.1. The decomposition of the exterior derivative. The formula (e) of Lemma III.3 leads to the decomposition of the exterior derivative operator on fibered manifolds in the following sense.

Proposition III.8. *Let $\rho \in \Omega^r_q W$ and $\eta \in \Omega^r_t W$ be differential forms. Then the exterior derivative* d *decomposes into two invariant parts*

$$(\pi^{r+2,r})^*\,\mathrm{d}\,\rho = \mathrm{d_H}\,\rho + \mathrm{d_C}\,\rho, \tag{III.24}$$

where

$$\mathrm{d_H}\,\rho = \sum_{k=0}^{q} \mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_k\,\rho \tag{III.25}$$

$$\mathrm{d_C}\,\rho = \sum_{k=0}^{q} \mathrm{p}_{k+1}\,\mathrm{d}\,\mathrm{p}_k\,\rho. \tag{III.26}$$

Thus, the following identities hold

(a) $\mathrm{d_H} \circ \mathrm{d_H} = 0$, $\mathrm{d_C} \circ \mathrm{d_C} = 0$ and $\mathrm{d_H} \circ \mathrm{d_C} = -\,\mathrm{d_C} \circ \mathrm{d_H}$,

(b) $\mathrm{d_H}(\rho \wedge \eta) = \mathrm{d_H}\,\rho \wedge (\pi^{r+2,r})^*\eta + (-1)^q(\pi^{r+2,r})^*\rho \wedge \mathrm{d_H}\,\eta$,

(c) $\mathrm{d_C}(\rho \wedge \eta) = \mathrm{d_C}\,\rho \wedge (\pi^{r+2,r})^*\eta + (-1)^q(\pi^{r+2,r})^*\rho \wedge \mathrm{d_C}\,\eta$.

Proof. (a) Using the identity (e) of Lemma III.3 and omitting the pullbacks by the natural jet projections we write

$$\begin{aligned}
\mathrm{d_H^2} &= \sum_{k=0}^{q} \mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_k \sum_{l=0}^{q} \mathrm{p}_l\,\mathrm{d}\,\mathrm{p}_l \\
&= \sum_{k=0}^{q} \mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_k \\
&= \sum_{k=0}^{q} (\mathrm{p}_k\,\mathrm{d} - \mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_{k-1})\,\mathrm{d}\,\mathrm{p}_k = 0
\end{aligned}$$

and similarly the analogous identity

$$\begin{aligned}
\mathrm{d_C^2} &= \sum_{k=0}^{q} \mathrm{p}_{k+1}\,\mathrm{d}\,\mathrm{p}_k \sum_{l=0}^{q} \mathrm{p}_{l+1}\,\mathrm{d}\,\mathrm{p}_l \\
&= \sum_{k=0}^{q} \mathrm{p}_{k+1}\,\mathrm{d}\,\mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_{k-1} \\
&= \sum_{k=0}^{q} (\mathrm{p}_{k+1}\,\mathrm{d} - \mathrm{p}_{k+1}\,\mathrm{d}\,\mathrm{p}_{k+1})\,\mathrm{d}\,\mathrm{p}_{k-1} = 0.
\end{aligned}$$

From

$$0 = \mathrm{d}^2 = \mathrm{d_H^2} + \mathrm{d_C^2} + \mathrm{d_H}\,\mathrm{d_C} + \mathrm{d_C}\,\mathrm{d_H}$$

we obtain the third identity.

(b) Without restricting the generality of the problem we can assume that $q \leq t$ and set $\mathrm{p}_k\,\rho = 0$ for $k < 0$ or $k > q$ and $\mathrm{p}_k\,\eta = 0$ for $k < 0$ and $k > t$.

$$\begin{aligned}
\mathrm{d_H}(\rho \wedge \eta) &= \sum_{k=0}^{q+t} \mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_k(\rho \wedge \eta) \\
&= \sum_{k=0}^{q+t} \mathrm{p}_k\,\mathrm{d} \sum_{l=0}^{q} \mathrm{p}_l\,\rho \wedge \mathrm{p}_{k-l}\,\eta \\
&= \sum_{k=0}^{q+t} \mathrm{p}_k \sum_{l=0}^{q} \left[\mathrm{d}\,\mathrm{p}_l\,\rho \wedge \mathrm{p}_{k-l}\,\eta + (-1)^q\,\mathrm{p}_l\,\rho \wedge \mathrm{d}\,\mathrm{p}_{k-l}\,\eta\right] \\
&= \sum_{k=0}^{q+t} \mathrm{p}_k \sum_{l=0}^{q} [\mathrm{p}_l\,\mathrm{d}\,\mathrm{p}_l\,\rho \wedge \mathrm{p}_{k-l}\,\eta + \mathrm{p}_{l+1}\,d\,\mathrm{p}_l\,\rho \wedge \mathrm{p}_{k-l}\,\eta + \\
&\quad + (-1)^q\,\mathrm{p}_l\,\rho \wedge \mathrm{p}_{k-l}\,\mathrm{d}\,\mathrm{p}_{k-l}\,\eta + (-1)^q\,\mathrm{p}_l\,\rho \wedge \mathrm{p}_{k-l+1}\,\mathrm{d}\,\mathrm{p}_{k-l}\,\eta]
\end{aligned}$$

and since $\mathrm{p}_k(\mathrm{p}_{l+1}\,\mathrm{d}\,\mathrm{p}_l\,\rho \wedge \mathrm{p}_{k-l}\,\eta) = 0$, $\mathrm{p}_k(\mathrm{p}_l\,\rho \wedge \mathrm{p}_{k-l+1}\,\mathrm{d}\,\mathrm{p}_{k-l}\,\eta) = 0$ and the remaining summands are k-contact

$$\begin{aligned}
&= \sum_{k=0}^{q+t} \sum_{l=0}^{q} [\mathrm{p}_l\,\mathrm{d}\,\mathrm{p}_l\,\rho \wedge \mathrm{p}_{k-l}\,\eta + (-1)^q\,\mathrm{p}_l\,\rho \wedge \mathrm{p}_{k-l}\,\mathrm{d}\,\mathrm{p}_{k-l}\,\eta] \\
&= \sum_{u=0}^{t} \sum_{l=0}^{q} [\mathrm{p}_l\,\mathrm{d}\,\mathrm{p}_l\,\rho \wedge \mathrm{p}_u\,\eta + (-1)^q\,\mathrm{p}_l\,\rho \wedge \mathrm{p}_u\,\mathrm{d}\,\mathrm{p}_u\,\eta] \\
&= \mathrm{d_H}\,\rho \wedge \eta + (-1)^q \rho \wedge \mathrm{d_H}\,\eta.
\end{aligned}$$

The formula (c) is obtained from $\mathrm{d_C} = (\pi^{r+2,r})^*\,\mathrm{d} - \mathrm{d_H}$. □

3.2. Formal derivatives of differential forms. There are several equivalent ways to define formal derivatives of differential forms. We strive for the most natural definition possible. We notice that from the previous paragraph we have that $\mathrm{d}\,\mathrm{d_H}\,\rho = -\,\mathrm{d_H}\,\mathrm{d}\,\rho$ for any differential form $\rho \in \Omega^r_q W$. We would like to split the differential form $\mathrm{d_H}\,\rho$ (this form always contains at least one product term $\mathrm{d}\,x^i$ in every summand) into

(III.27) $$\mathrm{d_H}\,\rho = (-1)^q\,\mathrm{D}_i\,\rho \wedge \mathrm{d}\,x^i.$$

Such a decomposition of the exterior product is, however, generally ambiguous. This can be seen from the identity

(III.28) $$\mathscr{D}_i \rho \wedge \mathrm{d}\,x^i = \mathrm{D}_i\,\rho \wedge \mathrm{d}\,x^i,$$

where

$$\mathscr{D}_i \rho = \mathrm{D}_i\,\rho + \varphi_i \delta_{ij} \wedge \mathrm{d}\,x^j,$$

and φ_i is any system of $(q-1)$-forms. Therefore it seems more appropriate to define the formal derivative axiomatically and then prove the equation (III.27).

Recall that the exterior derivative of functions has already been defined by (II.12). Let $\rho \in \Omega^r_q W$, $q > 0$, be a differential q-form. We define $\mathrm{D}_i\,\rho$, the *formal derivative* of a differential form ρ with respect to the coordinate x^i, by demanding that it commutes with exterior differentiation, so that

$$\mathrm{D}_i\,\mathrm{d}\,\rho = \mathrm{d}\,\mathrm{D}_i\,\rho \tag{III.29}$$

and, if $\eta \in \Omega^r_t W$ is another differential form, by obeying the Leibniz rule

$$\mathrm{D}_i(\rho \wedge \eta) = \mathrm{D}_i\,\rho \wedge \eta + \rho \wedge \mathrm{D}_i\,\eta. \tag{III.30}$$

It is clear that the formal derivative given by the properties (II.12), (III.29) and (III.30) is defined correctly.

Lemma III.9. *Let $\rho \in \Omega^r_q W$ be a differential form. Then*

$$\mathrm{d_H}\,\rho = (-1)^q\,\mathrm{D}_i\,\rho \wedge \mathrm{d}\,x^i. \tag{III.31}$$

Proof. The formal derivative for functions was defined in (II.12). First, we shall compute the formal derivatives of 1-forms.

$$\begin{aligned} \mathrm{D}_i\,\mathrm{d}\,x^j &= \mathrm{d}\,\mathrm{D}_i\,x^j = \mathrm{d}\,\delta^j_i = 0 \\ \mathrm{D}_i\,\mathrm{d}\,y^\sigma_J &= \mathrm{d}\,\mathrm{D}_i\,y^\sigma_J = \mathrm{d}\,y^\sigma_{Ji} \\ \mathrm{D}_i\,\omega^\sigma_J &= \mathrm{D}_i(\mathrm{d}\,y^\sigma_J - y^\sigma_{Jj}\,\mathrm{d}\,x^j) \\ &= \mathrm{d}\,y^\sigma_{Ji} - y^\sigma_{Jji}\,\mathrm{d}\,x^j \\ &= \mathrm{d}\,y^\sigma_{Ji} - y^\sigma_{Jij}\,\mathrm{d}\,x^j \\ &= \omega^\sigma_{Ji} \end{aligned} \tag{III.32}$$

Second, for each k-contact component in the contact decomposition of the differential form ρ, given in local coordinates by

$$\mathrm{p}_k\,\rho = B^{J_1\dots J_k}_{\sigma_1\dots\sigma_k\;i_{k+1}\dots i_q}\omega^{\sigma_1}_{J_1} \wedge \ldots \wedge \omega^{\sigma_k}_{J_k} \wedge \mathrm{d}\,x^{i_{k+1}} \wedge \ldots \wedge \mathrm{d}\,x^{i_q}$$

we compute its formal derivative using the Leibniz rule (III.30), obtaining

$$\begin{aligned} \mathrm{D}_i\,\rho = {} & \mathrm{D}_i\,B^{J_1\dots J_k}_{\sigma_1\dots\sigma_k\;i_{k+1}\dots i_q}\omega^{\sigma_1}_{J_1} \wedge \ldots \wedge \omega^{\sigma_k}_{J_k} \wedge \mathrm{d}\,x^{i_{k+1}} \wedge \ldots \wedge \mathrm{d}\,x^{i_q} + \\ & + kB^{J_1\dots J_k}_{\sigma_1\dots\sigma_k\;i_{k+1}\dots i_q}\omega^{\sigma_1}_{J_1 i} \wedge \omega^{\sigma_2}_{J_2} \wedge \ldots \wedge \omega^{\sigma_k}_{J_k} \wedge \mathrm{d}\,x^{i_{k+1}} \wedge \ldots \wedge \mathrm{d}\,x^{i_q}. \end{aligned} \tag{III.33}$$

On the other hand we can compute

$$\begin{aligned} \mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_k\,\rho = {} & \mathrm{h}\,\mathrm{d}\,B^{J_1\dots J_k}_{\sigma_1\dots\sigma_k\;i_{k+1}\dots i_q}\omega^{\sigma_1}_{J_1} \wedge \ldots \wedge \omega^{\sigma_k}_{J_k} \wedge \mathrm{d}\,x^{i_{k+1}} \wedge \ldots \wedge \mathrm{d}\,x^{i_q} + \\ & + kB^{J_1\dots J_k}_{\sigma_1\dots\sigma_k\;i_{k+1}\dots i_q}\mathrm{p}_1\,\mathrm{d}\,\omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_2}_{J_2} \wedge \ldots \wedge \omega^{\sigma_k}_{J_k} \wedge \mathrm{d}\,x^{i_{k+1}} \wedge \ldots \wedge \mathrm{d}\,x^{i_q}. \end{aligned} \tag{III.34}$$

From (III.33) and (III.34) we conclude that (III.31) indeed holds. □

Based on the previous Lemma (III.9), we now formulate the properties of formal derivatives.

Lemma III.10. *Let (V, ψ) be a fibered chart on Y and let $\rho \in \Omega^r_q W$ be a differential q-form. Then the following holds*

(a) If we denote by bars another coordinate system such that $V \cap \bar{V} \neq \emptyset$ and denote by $\bar{\mathrm{D}}_j$ the formal derivative with respect to the coordinate $\bar{x}^j$, we obtain the following transformation rule

$$\mathrm{D}_i\, \rho = \frac{\partial \bar{x}^j}{\partial x^i} \bar{\mathrm{D}}_j \rho. \tag{III.35}$$

(b) The formal derivatives commute, so that

$$\mathrm{D}_i\, \mathrm{D}_j\, \rho = \mathrm{D}_j\, \mathrm{D}_i\, \rho. \tag{III.36}$$

Proof. (a) The transformation formula is the direct consequence of the previous Lemma. We have

$$(-1)^q\, \mathrm{D}_i\, \rho \wedge \mathrm{d}\, x^i = \mathrm{d_H}\, \rho = (-1)^q \bar{\mathrm{D}}_j \rho \wedge \mathrm{d}\, \bar{x}^j$$

$$\mathrm{D}_i\, \rho \wedge \mathrm{d}\, x^i = \bar{\mathrm{D}}_j \rho \wedge \frac{\partial \bar{x}^j}{\partial x^i}\, \mathrm{d}\, x^i$$

$$\mathrm{D}_i\, \rho = \frac{\partial \bar{x}^j}{\partial x^i} \bar{\mathrm{D}}_j \rho$$

(b) First, we check the property (III.36) for functions and basis 1-forms. For functions we obtain

$$\mathrm{D}_i\, f = \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial y^\sigma_J} y^\sigma_{Ji}$$

$$\mathrm{D}_j\, \mathrm{D}_i\, f = \frac{\partial^2 f}{\partial x^j \partial x^i} + \frac{\partial^2 f}{\partial y^\sigma_J \partial x^j} y^\sigma_{Ji} + \frac{\partial^2 f}{\partial y^\nu_K \partial x^i} y^\nu_{Kj} + \frac{\partial^2 f}{\partial y^\sigma_J \partial y^\nu_K} y^\sigma_{Ji} y^\nu_{Kj} + \frac{\partial f}{\partial y^\sigma_J} y^\sigma_{Jij}$$

$$\mathrm{D}_j\, \mathrm{D}_i\, f = \mathrm{D}_i\, \mathrm{D}_j\, f$$

For the 1-form $\mathrm{d}\, x^j$ we get the result trivially and for the contact 1-form we conclude

$$\mathrm{D}_i\, \mathrm{D}_j\, \omega^\sigma_J = \omega^\sigma_{Jji} = \omega^\sigma_{Jij} = \mathrm{D}_j\, \mathrm{D}_i\, \omega^\sigma_J.$$

Let $\eta \in \Omega^r_t W$ be another t-form. Using the Leibniz rule (III.30) we obtain

$$\begin{aligned} \mathrm{D}_j\, \mathrm{D}_i(\rho \wedge \eta) &= \mathrm{D}_j\, (\mathrm{D}_i\, \rho \wedge \eta + \rho \wedge \mathrm{D}_i\, \eta) \\ &= \mathrm{D}_j\, \mathrm{D}_i\, \rho \wedge \eta + \rho \wedge \mathrm{D}_j\, \mathrm{D}_i\, \eta + \mathrm{D}_i\, \rho \wedge \mathrm{D}_j\, \eta + \mathrm{D}_j\, \rho \wedge \mathrm{D}_i\, \eta \end{aligned}$$

The last two terms represent an expression symmetric in i and j. Thus

$$\mathrm{D}_j\, \mathrm{D}_i(\rho \wedge \eta) - \mathrm{D}_i\, \mathrm{D}_j(\rho \wedge \eta) = (\mathrm{D}_j\, \mathrm{D}_i\, \rho - \mathrm{D}_i\, \mathrm{D}_j\, \rho) \wedge \eta + \rho \wedge (\mathrm{D}_j\, \mathrm{D}_i\, \eta - \mathrm{D}_i\, \mathrm{D}_j\, \eta)$$

and as any formal derivative is built from formal derivatives of functions and the 1-forms $\mathrm{d}\, x^i$ and ω^σ_J for which the formal derivatives commute, we can set $\mathrm{D}_i \,\mathrm{D}_j = \mathrm{D}_j \,\mathrm{D}_i$ in general. □

Remark 6. Since the formal derivatives commute, we shall use the notation

$$\mathrm{D}_J = \mathrm{D}_{j_s} \dots \mathrm{D}_{j_1},$$

where $J = (j_1 \dots j_s)$, from this point onward.

We shall now prove some further properties of the various geometric operations which pertain to contact components of differential forms. These will be needed in the next chapter.

Theorem III.11. *Let Ξ be a π-vertical vector field on Y and $\rho \in \Omega^r_q W$ a differential q-form on $J^r Y$. Then the following hold:*

(III.37) $$J^{r+2}\Xi \lrcorner \mathrm{p}_k \,\mathrm{d}\, \mathrm{p}_k \,\rho = -\,\mathrm{p}_{k-1}\,\mathrm{d}\left(J^{r+1}\Xi \lrcorner \mathrm{p}_k \,\rho\right)$$

$$\mathscr{L}_{J^{r+2}\Xi}\left(\pi^{r+2,r+1}\right)^* \mathrm{p}_k \,\rho = J^{r+2}\Xi \lrcorner \mathrm{p}_{k+1}\,\mathrm{d}\,\mathrm{p}_k\,\rho + \mathrm{p}_k\,\mathrm{d}\left(J^{r+1}\Xi \lrcorner \mathrm{p}_k\,\rho\right)$$

Proof. We shall verify that the Lie derivative by the prolongation of a π-vertical vector field does not change the order of contactness of a differential form. It suffices to prove this locally for the generators. The statement is trivial for functions, for the basis 1-forms we get

$$\begin{aligned} J^r\Xi \lrcorner \omega^\nu_I &= \sum_{|J|=0}^{r}\left(\mathrm{D}_J\,\Xi^\sigma \frac{\partial}{\partial y^\sigma_J}\right) \lrcorner \omega^\nu_I = \mathrm{D}_I\,\Xi^\nu \\ (\pi^{r+2,r+1})^* \mathscr{L}_{J^r\Xi}\omega^\nu_I &= -\,\mathrm{d}\,x^j\,\mathrm{D}_{Ij}\,\Xi^\nu + \mathrm{d}\,x^j\,\mathrm{D}_{Ij}\,\Xi^\nu + \omega^\sigma_J \frac{\partial \Xi^\nu_I}{\partial y^\sigma_J} \\ J^r\Xi \lrcorner \mathrm{d}\,x^j &= 0, \end{aligned}$$

using the well-known Cartan formula

(III.38) $$\mathscr{L}_\xi = \xi \lrcorner \mathrm{d}\,\rho + \mathrm{d}\,\xi \lrcorner \rho,$$

valid for any vector field ξ and differential form ρ on any smooth manifold. Notice that it holds

(III.39) $$\left(\pi^{r+2,r+1}\right)^* \mathrm{d}\,\mathrm{p}_k\,\rho = \mathrm{p}_{k+1}\,\mathrm{d}\,\mathrm{p}_k\,\rho + \mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_k\,\rho$$

and for a π-vertical Ξ, following Lemma III.3 (b),

$$J^{r+1}\Xi \lrcorner \mathrm{p}_k\,\rho = \mathrm{p}_{k-1}(\mathrm{p}\,J^r\Xi) \lrcorner \rho$$

Since Ξ is π-vertical $J^{r+1}\Xi \lrcorner \, \mathrm{p}_k \, \rho$ is $(k-1)$-contact. Using the Cartan formula and the formulas above we obtain

$$
\begin{aligned}
&\mathscr{L}_{J^{r+2}\Xi} \left(\pi^{r+2,r+1}\right)^* \mathrm{p}_k \, \rho = \\
&\qquad = J^{r+2}\Xi \lrcorner \, \mathrm{d} \left(\pi^{r+2,r+1}\right)^* \mathrm{p}_k \, \rho + \mathrm{d} \left(J^{r+2}\Xi \lrcorner \left(\pi^{r+2,r+1}\right)^* \mathrm{p}_k \, \rho\right) = \\
&\qquad = J^{r+2}\Xi \lrcorner \left(\mathrm{p}_{k+1} \, \mathrm{d} \, \mathrm{p}_k \, \rho + \mathrm{p}_k \, \mathrm{d} \, \mathrm{p}_k \, \rho\right) + \left(\mathrm{p}_{k-1} \, \mathrm{d} + \mathrm{p}_k \, \mathrm{d}\right) \left(J^{r+1}\Xi \lrcorner \, \mathrm{p}_k \, \rho\right).
\end{aligned}
$$

This gives the required formula as follows: The contraction of an k-contact q-form by the prolongation of a π-vertical vector field $J^r\Xi$ leads to an k-contact $(q-1)$-form and therefore, the formula can be split into its k-contact and $(k+1)$-contact parts, which give the required formulas. $\square$

Chapter IV

Formal Differential Operators

We shall recall the notion of a differential operator as understood in [35] and [19]. We then specify what will be meant by formal differential operators. Then we shall focus our attention on certain special formal differential operators which will be useful for the representation of the variational sequence by differential forms and the calculus of variations on fibered manifolds. This constitutes some generalisations of [4] to the case of finite order jet prolongations of fibered manifolds.

1. Formal Differential Operators

1.1. Differential operators. Let $\tau\colon W \to X$ and $\rho\colon Z \to X$ be two fibered manifolds over the same base X and let $\sec\tau$ (resp. $\sec\rho$) denote the set of smooth local sections of τ (resp. ρ). A mapping $D\colon \sec\tau \to \sec\rho$ is said to be a *differential operator*, if there exist an integer $r \geq 0$ and a morphism of fibered manifolds $D^r\colon J^rW \to Z$ over the identity id_X such that for every section $\gamma \in \sec\tau$

$$D(\gamma) = D^r \circ J^r\gamma. \qquad \text{(IV.1)}$$

If there exists such an integer r then for every $s \geq r$ we may construct the morphism $D^s\colon J^sW \to Z$ over the identity id_X by taking $D^s = D^r \circ \tau^{s,r}$, where $\tau^{s,r}\colon J^sW \to J^rW$ is the canonical projection. The smallest integer r for which there exists a morphism D^r satisfying (IV.1) for all sections $\gamma \in \sec\tau$ is called the *order* of the differential operator D. The situation is depicted with the Figure 1.

$$\begin{tikzcd} J^rW \arrow[d] \arrow[dr, "D^r"] \arrow[dd, bend right=60, "\tau^r"'] & \\ W \arrow[d, "\tau"] & Z \arrow[dl, "\rho"] \\ X & \end{tikzcd} \qquad \begin{tikzcd} J^rW \arrow[d] \arrow[dr, "D^r"] & \\ W & Z \\ X \arrow[u, "\gamma"] \arrow[ur, "D^r\circ J^r\gamma"'] \arrow[uu, bend left=60, "J^r\gamma"] & \end{tikzcd}$$

Figure 1. Differential operators

Every morphism of fibered manifolds $\alpha\colon J^rW \to W$ over the identity id_X defines a differential operator $D_\alpha\colon \sec\tau \to \sec\rho$ by the formula

$$D_\alpha(\gamma) = \alpha \circ J^r\gamma. \tag{IV.2}$$

Let $\tau\colon W \to X$ be a fibered manifold with base X and let $\rho\colon Z \to W$ be a fibered manifold with base W. Denote by $\sec\tau$ (resp. $\sec\tau\circ\rho$) the set of smooth local sections of τ (resp. $\tau\circ\rho$). A differential operator $D\colon \sec\tau \to \sec\tau\rho$ is said to be a *prolongation operator*, if for every $\gamma \in \sec\tau$

$$\rho \circ D(\gamma) = \gamma. \tag{IV.3}$$

We can also describe prolongation operators by requiring that the diagram in Figure 2 commutes.

$$\begin{tikzcd} J^rW \arrow[d] \arrow[dr, "D^r"] & \\ W & Z \arrow[l, "\rho"'] \\ X \arrow[u, "\gamma"] \arrow[uu, bend left=60, "J^r\gamma"] & \end{tikzcd}$$

Figure 2. Prolongation operators

$$\begin{tikzcd} TY \arrow[r, "T\pi"] \arrow[d, "p"'] \arrow[dr, "\pi\circ p"] & TX \arrow[d, "p"] \\ Y \arrow[r, "\pi"'] & X \end{tikzcd}$$

Figure 3. The tangent bundles of fibered manifolds

1.2. Formal differential operators. Let $\pi\colon Y \to X$ be a fibered manifold. We shall denote the foot projection in the tangent bundle of any manifold X by $p\colon TX \to X$. Let us consider the scheme in Figure 3. The triple $(TY, \pi \circ p, X)$ is clearly a vector bundle. We shall take the subbundle $(VY, \pi \circ p, X)$ of vertical vectors (see Paragraph 1.3 in Chapter II) and consider differential operators applied to local sections of this subbundle.

It is essential that in the case of vertical bundles, there exists a canonical isomorphism that enables to identify $VJ^rY \cong J^rVY$. More generally, there exists a canonical fibered morphism $t^r\colon J^rTY \to TJ^rY$. We denote

- (x^i, y^σ) adapted coordinates on the fibered manifold $Y \to X$,
- (x^i, y^σ_J), where $0 \le |J| \le r$, adapted coordinates on the composite fibered manifold $J^rY \to \cdots \to Y \to X$,
- $(x^i, y^\sigma; \xi^i, \Xi^\sigma)$ adapted coordinates on the composite fibered manifold $TY \to TX \to X$,
- $(x^i, y^\sigma_J; \xi^i, \Xi^\sigma_J)$, where $0 \le |J| \le r$, adapted coordinates on the composite fibered manifold $TJ^rY \to J^rY \to X$ and
- $(x^i, y^\sigma_J, \zeta^i_J, \Upsilon^\sigma_J)$ adapted coordinates on the composite fibered manifold $J^rTY \to TY \to X$.

The canonical fibered isomorphism between VJ^rY and J^rVY is given simply by $\Xi^\sigma_J = \Upsilon^\sigma_J$ and the fibered morphism $t^r\colon J^rTY \to TJ^rY$ is given by using the morphism $t\colon JTY \to TJY$ which is expressed by

$$\xi^i = \zeta^i$$
$$\Xi^\sigma_i = \Upsilon^\sigma_i - y^\sigma_j \zeta^j_i.$$

The local sections of the vertical subbundle are local *vertical vector fields* on Y. We shall denote the set of vertical vector fields on a neighborhood W, where $W \subset Y$ is an open set, by $\mathscr{V}W$. Differential operators P of the type

$$P : \mathscr{V}W \ni \Xi \to P(\Xi) = D^r \circ J^r\Xi \in \sec \rho$$

are said to be *formal differential operators*. The corresponding diagrammatic description focusing on the domain of definition is drawn in Figure 4.

$$\begin{tikzcd}
& J^rVY \cong VJ^rY \arrow[d] \arrow[dr, "D^r"] & \\
TY \arrow[d, "T\pi"'] & VY \arrow[l, hook'] \arrow[d, "T\pi"'] & Z \arrow[ddl, "\rho"'] \\
TX \arrow[d, "p"'] & 0_{TX} \arrow[l, hook'] \arrow[d, "p"'] & \\
X & X &
\end{tikzcd}$$

Figure 4. Formal differential operators

1.3. The codomains of formal differential operators. Next we specify the structure of the fibered manifold $Z \to X$, whose sections serve as targets for differential operators. For our purposes and from this point onward, the fibered manifold Z will *always* be the bundle of antisymmetric $(q, 0)$-tensors over J^rY. We shall denote this bundle by $\Lambda_q T^* J^r Y$. This is clearly a composite fibered manifold $\rho : \Lambda_q T^* J^r Y \to J^r Y \to X$ with respect to the projection ρ. Also, from this point onward, we shall consider the mapping D^r in the sense of a morphism of vector bundles $VJ^rY \to \Lambda_q T^* J^r Y$. The local sections of the bundle $\Lambda_q T^* J^r Y$ are differential q-forms. We denote the set of local sections on the neighborhood $W^r = (\pi^{r,0})^{-1}(W)$, where $W \subset Y$ is an open set, by $\Omega_q^r W$. Note that such sets of local sections have the structure of modules over the ring of functions on W^r.

Example 2. We shall make use of the following formal differential operator. It is a natural differential operator. Let $\rho \in \Omega_{q+1}^r Y$ be a given differential $(q+1)$-form. We shall denote its values in $\Lambda_q T^* J^r Y$ by the same symbol ρ. Consider the vector bundle morphism D^r given by

$$D^r \colon J^rVY \cong VJ^rY \ni \Upsilon \to \Upsilon \lrcorner \rho \in \Lambda_q T^* J^r Y. \tag{IV.4}$$

Then the mapping given by

$$C_\rho \colon \mathscr{V} W \ni \Xi \to D^r \circ J^r \Xi = J^r \Xi \lrcorner \rho \in \Omega_q^r W \tag{IV.5}$$

is a formal differential operator. Natural differential operators are treated in more detail and generality in [35] and [19].

1.4. Local expressions of formal differential operators. We shall derive the local expression of formal differential operators. In the previous paragraphs we saw that the r-jet prolongation of the vertical bundle is isomorphic to the vertical bundle of the r-jet prolongation or $J^rVY \cong VJ^rY$. Let $W \subset Y$ be an open set and $\Xi \in \mathscr{V} W$ be a vertical vector field given in local coordinates by

$$\Xi = \Xi^\sigma \frac{\partial}{\partial y^\sigma}. \tag{IV.6}$$

Then its r-jet prolongation is given by

$$J^r \Xi = (\mathrm{D}_J \, \Xi^\sigma) \frac{\partial}{\partial y^\sigma_J}. \tag{IV.7}$$

We obtain the local expression of a formal differential operator P as

$$P(\Xi) = \sum_{|J|=0}^{r} (\mathrm{D}_J \, \Xi^\sigma) P^J_\sigma , \tag{IV.8}$$

where the coefficients P^J_σ are smooth differential q-forms. Since we know how the formal derivative operates on differential forms, we attempt to move the formal derivatives from the components of the vertical vector field to the differential q-forms P^J_σ by using a repeated "integration by parts" procedure. This is the heart of the following lemma.

Lemma IV.1. *Any formal differential operator P can be uniquely rewritten as*

$$P(\Xi) = \sum_{|I|=0}^{r} \mathrm{D}_I \left(\Xi^\sigma Q^I_\sigma \right) , \tag{IV.9}$$

where

$$Q^I_\sigma = \sum_{|J|=0}^{r-|I|} \tbinom{|I|+|J|}{|I|} (-1)^{|J|} \, \mathrm{D}_J \, P^{IJ}_\sigma . \tag{IV.10}$$

Proof. To establish this we simply expand the equation (IV.9) using the product rule (III.30) and obtain

$$P(\Xi) = \sum_{|J|=0}^{r} \mathrm{D}_J (\Xi^\sigma) \sum_{|I|=0}^{r-|J|} \tbinom{|I|+|J|}{|J|} \, \mathrm{D}_I \, Q^{JI}_\sigma .$$

We shall prove that

$$P^J_\sigma = \sum_{|I|=0}^{r-|J|} \tbinom{|I|+|J|}{|I|} \, \mathrm{D}_I \, Q^{JI}_\sigma . \tag{IV.11}$$

For given P^J_σ this system of equations has the unique solution

$$\begin{aligned} P^{J_r}_\sigma &= Q^{J_r}_\sigma \\ P^{J_{r-1}}_\sigma &= Q^{J_{r-1}}_\sigma + r\,\mathrm{D}_j\, Q^{J_{r-1}j}_\sigma \\ \vdots &= \qquad \vdots \\ P_\sigma &= Q_\sigma + \qquad \dots \qquad + \mathrm{D}_{J_r}\, Q^{J_r}_\sigma . \end{aligned}$$

For clarity we index the multiindices by their lengths. It is obvious that $Q^{I_p}_\sigma = 0$ for $0 \le p \le r$ implies $P^{J_t}_\sigma = 0$ for $0 \le t \le r$. Eliminating the forms Q^I_σ from the above written system recursively for $I = J_r, \dots, J_{p+1}$ we obtain (IV.10). It remains to verify whether Q^I_σ satisfy (IV.11) identically.

$$\begin{aligned} P^{J_p}_\sigma &= \sum_{q=0}^{r-p} \tbinom{p+q}{q}\, \mathrm{D}_{I_q} \left[\sum_{s=0}^{r-p-q} \tbinom{p+q+s}{p+q} (-\,\mathrm{D})_{K_s} P^{I_p J_{q-s} K_s}_\sigma \right] \\ &= \sum_{t=0}^{r-p} \left[\sum_{q=0}^{t} (-1)^{t-q} \tbinom{p+q}{p} \tbinom{p+t}{p+q} \right] \mathrm{D}_{I_t}\, P^{J_r I_t}_\sigma . \end{aligned}$$

The combinatorial sum in the square brackets is equal to 1 for $t = 0$ and to 0 otherwise. We can prove this by induction on p and t. □

The coefficients Q^J_σ, where $|J| = r$ form the so called *symbol* of the formal differential operator P. In general, the remaining coefficients Q^J_σ, where $1 \le |J| < r$, don't have any geometric meaning. In the next section we shall investigate the properties of the coefficients Q_σ and formulate conditions under which these coefficients give rise to formal differential operators of order zero.

2. Euler operators

2.1. Euler operators. First we shall present two examples of some simple formal differential operators and in the second example compute the transformation properties of the coefficients Q_σ arising from the Lemma IV.1.

Example 3. In this example the indices k and l (as well as the usual reserved indices i and j) are summed over 1 to n. Let us take a form $\rho \in \Omega^1_3 W$ and take its 2-contact component. In local coordinates we shall write

$$\mathrm{p}_2\, \rho = A_{\sigma\nu\; i}\omega^\sigma \wedge \omega^\nu \wedge \mathrm{d}\, x^i + 2A^j_{\sigma\nu\; i}\omega^\sigma_j \wedge \omega^\nu \wedge \mathrm{d}\, x^i + A^{jk}_{\sigma\nu\; i}\omega^\sigma_j \wedge \omega^\nu_k \wedge \mathrm{d}\, x^i ,$$

and similarly for the vertical vector field Ξ on Y

$$\Xi = \Xi^\sigma \partial_\sigma \,, \quad J^1\Xi = \Xi^\sigma \partial_\sigma + \Xi^\sigma_j \partial^j_\sigma = \Xi^\sigma \partial_\sigma + \mathrm{D}_j\, \Xi^\sigma \partial^j_\sigma .$$

By means of the differential form $\mathrm{p}_2\,\rho$ we shall construct the formal differential operator given by (IV.5). In coordinates we obtain

$$P(\Xi) = \Xi^\sigma \left(A_{\sigma\nu\; i} \omega^\nu \wedge \mathrm{d}\, x^i - A_{\nu\sigma\; i}^{\;\;k} \omega^\nu_k \wedge \mathrm{d}\, x^i \right) + \\ + \mathrm{D}_j\, \Xi^\sigma \left(A^j_{\sigma\nu\; i} \omega^\nu \wedge \mathrm{d}\, x^i + A^{jk}_{\sigma\nu\; i} \omega^\nu_k \wedge \mathrm{d}\, x^i \right).$$

We compute the formula for Q_σ.

$$\begin{aligned} Q_\sigma &= P_\sigma - \mathrm{D}_j\, P^j_\sigma \\ &= A_{\sigma\nu\; i}\omega^\nu \wedge \mathrm{d}\, x^i - A_{\nu\sigma\; i}^{\;\;k} \wedge \omega^\nu_k \wedge \mathrm{d}\, x^i - \\ &\quad - \mathrm{D}_j \left(A^j_{\sigma\nu\; i}\omega^\nu \wedge \mathrm{d}\, x^i + A^{jk}_{\sigma\nu\; i}\omega^\nu_k \wedge \mathrm{d}\, x^i \right) \\ &= \left(A_{\sigma\nu\; i} - \mathrm{D}_j\, A^j_{\sigma\nu\; i} \right) \omega^\nu \wedge \mathrm{d}\, x^i + \left(-A_{\nu\sigma\; i}^{\;\;k} - A^k_{\sigma\nu\; i} - \mathrm{D}_j\, A^{jk}_{\sigma\nu\; i} \right) \omega^\nu_k \wedge \mathrm{d}\, x^i - \\ &\quad - A^{jk}_{\sigma\nu\; i}\omega^\nu_{kj} \wedge \mathrm{d}\, x^i. \end{aligned}$$

We can easily observe that the zeroth order formal differential operator defined by the formula $Q(\Xi) = \Xi^\sigma Q_\sigma$ is not correctly defined. The situation changes if we consider $n = \dim X = 1$ or more generally, if we consider formal differential operators whose codomains are spaces of $(q-n)$ contact q-forms. This means that for formal differential operators which arise from contractions of given differential forms by r-order prolongations of vector fields we can consider only spaces of $(q-n+1)$-contact components.

The exact formulation of the above idea is given by the following Theorem.

Theorem IV.2. *Let $W \subset Y$ be an open set. Let $P : \mathscr{V}W \to \mathrm{p}_k\, \Omega^r_{n+k} W$, $n = \dim X$ be a formal differential operator of order r. Then there exists a unique formal differential operator of order zero*

$$Q : \mathscr{V}W \to \mathrm{p}_k\, \Omega^{2r}_{n+k} W,$$

such that

$$P(\Xi) = Q(\Xi) + \mathrm{p}_k\, \mathrm{d}\, \mathrm{p}_k\, R(\Xi) \tag{IV.12}$$

and locally

$$Q(\Xi) = Q_\sigma \Xi^\sigma, \tag{IV.13}$$

where Q_σ is defined in terms of P^J_σ as in (IV.10).

Q is called the *Euler operator* corresponding to P and R is a locally defined differential operator.

Proof. We proceed in two steps. First we prove the local existence of Q and then its uniqueness.

We start with the local existence of the decomposition (IV.12). As in (IV.9) we can write

$$\begin{aligned} P(\Xi) &= Q_\sigma \Xi^\sigma + \sum_{|J|=1}^{r} \mathrm{D}_J (\Xi^\sigma Q_\sigma^J) \\ &= Q_\sigma \Xi^\sigma + \mathrm{D}_i \sum_{|J|=0}^{r-1} \mathrm{D}_J (\Xi^\sigma Q_\sigma^{Ji}). \end{aligned}$$

The coefficients Q_σ^J are differential $(n+k)$-forms that can be written as

$$Q_\sigma^J = B_\sigma^J \wedge \omega_0 = (-1)^{n-1} B_\sigma^J \wedge \omega_j \wedge \mathrm{d}\, x^j$$

for arbitrary $j = 1, \ldots, n$, there is no summation over j and B_σ^J are some k-contact k-forms. We further define $\omega_j = \partial/\partial x^j \lrcorner\, \omega_0$. Then, taking into account that $\omega_j \wedge \mathrm{d}\, x^i = 0$ for $j \neq i$, we have

$$P(\Xi) = Q_\sigma \Xi^\sigma + \mathrm{D}_i \sum_{|J|=0}^{r-1} \mathrm{D}_J \left[\Xi^\sigma (-1)^{n-1} B_\sigma^{Jj} \wedge \omega_j \wedge \mathrm{d}\, x^i \right].$$

Using the product rule (III.30) we obtain

$$= Q_\sigma \Xi^\sigma + \mathrm{D}_i \sum_{|J|=0}^{r-1} \mathrm{D}_J \left[\Xi^\sigma (-1)^{n-1} B_\sigma^{Jj} \wedge \omega_j \right] \wedge \mathrm{d}\, x^i$$

and finally

$$P(\Xi) = Q_\sigma \Xi^\sigma + (-1)^{n-1} (-1)^{n+k-1}\, \mathrm{p}_k \, \mathrm{d}\, \mathrm{p}_k \sum_{|J|=0}^{r-1} (\mathrm{D}_J\, \Xi^\sigma B_\sigma^{Jj}) \wedge \omega_j,$$

the needed local decomposition, and the explicit expression for the differential operator R given by

$$R(\Xi) = (-1)^k \sum_{|J|=0}^{r-1} (\mathrm{D}_J\, \Xi^\sigma B_\sigma^{Jj}) \wedge \omega_j.$$

We shall turn to the proof of the uniqueness of Q. Suppose that Q is not given uniquely and

$$\begin{aligned} P(\Xi) &= Q(\Xi) + \mathrm{p}_k \, \mathrm{d}\, \mathrm{p}_k \, R(\Xi) \\ P(\Xi) &= \tilde{Q}(\Xi) + \mathrm{p}_k \, \mathrm{d}\, \mathrm{p}_k \, \tilde{R}(\Xi). \end{aligned}$$

Subtracting these equations we obtain

$$\Xi^\sigma M_\sigma = Q(\Xi) - \tilde{Q}(\Xi) = \mathrm{p}_k \,\mathrm{d}\,\mathrm{p}_k[\tilde{R}(\Xi) - R(\Xi)] = \mathrm{p}_k \,\mathrm{d}\,\mathrm{p}_k(R - \tilde{R})(\Xi).$$

Let Ξ_j, $j \in \{1, \ldots, k\}$ be arbitrary π-vertical vector fields on Y and let us form the expression

$$J^{r+2}\Xi_k \lrcorner \ldots \lrcorner J^{r+2}\Xi_1 \lrcorner \mathrm{p}_k \,\mathrm{d}\,\mathrm{p}_k(R - \tilde{R})(\Xi)$$

We apply (III.37) and arrive at

$$\Xi^\sigma N_\sigma = (-1)^k \,\mathrm{h}\,\mathrm{d}[J^{r+1}\Xi_k \lrcorner \ldots \lrcorner J^{r+1}\Xi_1 \lrcorner \mathrm{p}_k(R - \tilde{R})(\Xi)].$$

with

$$N_\sigma = J^{r+2}\Xi_k \lrcorner \ldots \lrcorner J^{r+2}\Xi_1 \lrcorner M_\sigma.$$

Choose an arbitrary piece $\Omega \subset U \subset X$ in the coordinate neighborhood and pick Ξ_j so that $\operatorname{supp} \Xi_j \subset \Omega$, and a local section γ of Y. Form the integral

$$\int_\Omega (J^{r+2}\gamma)^* \Xi^\sigma N_\sigma.$$

Due to Stokes' theorem and the fact that $\operatorname{supp} \Xi_j \subset \Omega$ this integral is zero. The section γ was arbitrary so $N_\sigma = 0$. Finally, the vector fields $\Xi_1, \ldots, \Xi_k$ were arbitrary so $M_\sigma = 0$ and $Q(\Xi) = \tilde{Q}(\Xi)$.

By means of partitions of unity we can construct globally defined operators, uniqueness assures that there is only one such operator. □

We can now turn to investigate, which zeroth order operators pertain to contractions of a given differential form by prolongations of vertical vector fields.

2.2. Contraction Euler operators. The contraction Euler operator shall be defined as the Euler operator corresponding to the formal differential operator defined by $J^r\Xi \lrcorner \mathrm{p}_k \rho$, where $\rho \in \Omega^r_{n+k}W$, for $k \geq 1$, is a given differential form. We shall express this operator in local coordinates.

The form $\mathrm{p}_k \rho$ is expressed in a given fibered chart as follows

$$\mathrm{p}_k \rho = B^{J_1 \ldots J_k}_{\sigma_1 \ldots \sigma_k} \omega^{\sigma_1}_{J_1} \wedge \ldots \wedge \omega^{\sigma_k}_{J_k} \wedge \omega_0.$$

The formal differential operator defined by the contraction is given by

$$C_{\mathrm{p}_k \rho}(\Xi) = J^{r+1}\Xi \lrcorner \mathrm{p}_k \rho = \sum_{|J|=0}^{r} (\mathrm{D}_J \Xi^\sigma)(\partial^J_\sigma \lrcorner \mathrm{p}_k \rho) \tag{IV.14}$$

as can be seen from Example 2. Using the Theorem IV.2 we can find the corresponding zero order operator I, called the *contraction Euler operator*, which is given locally by

$$C(\Xi) = \Xi^\sigma C_\sigma, \tag{IV.15}$$

where the coefficients C_σ are given by (IV.10) as

$$C_\sigma = \sum_{|J|=0}^{r} (-1)^{|J|} \,\mathrm{D}_J(\partial_\sigma^J \lrcorner\, \mathrm{p}_k\, \rho). \tag{IV.16}$$

We can use the contraction Euler operator to construct a new $\mathbb{R}$-linear mapping

$$I\colon \Omega^r_{n+k} W \to \Omega^{2r+1}_{n+k} W \tag{IV.17}$$

defined by

$$I(\rho) = \tfrac{1}{k}\omega^\sigma \wedge C_\sigma = \tfrac{1}{k}\omega^\sigma \wedge \sum_{|I|=0}^{r} (-1)^{|I|} \,\mathrm{D}_I(\partial_\sigma^I \lrcorner\, \mathrm{p}_k\, \rho) \tag{IV.18}$$

We shall call this mapping I the *contraction Euler mapping*. Since we have defined I in local coordinates, we need to verify that it is globally well defined. We shall check the transformation properties of the mapping I as defined by (IV.18). To this end we consider two fibered charts on Y, (V,ψ), $\psi = (x^i, y^\sigma)$ and $(\bar V, \bar\psi)$, $\bar\psi = (\bar x^i, \bar y^\sigma)$ such that $V \cap \bar V \neq \emptyset$. We shall equip all quantities expressed in the coordinate system $(\bar V, \bar\psi)$ by bars and according to (IV.15) we write

$$C(\Xi) = \Xi^\sigma C_\sigma = \bar\Xi^\nu \bar C_\nu,$$

where

$$\Xi = \Xi^\sigma \partial_\sigma = \bar\Xi^\nu \bar\partial_\nu,$$

and clearly it holds

$$\partial_\sigma = \frac{\partial}{\partial y^\sigma} = \frac{\partial \bar y^\nu}{\partial y^\sigma}\frac{\partial}{\partial \bar y^\nu} = \frac{\partial \bar y^\nu}{\partial y^\sigma}\bar\partial_\nu.$$

Therefore,

$$C_\sigma = \frac{\partial \bar y^\nu}{\partial y^\sigma}\bar C_\nu.$$

On the other hand, we have

$$\begin{aligned}
\omega^\sigma &= \mathrm{d}\, y^\sigma - y^\sigma_j \,\mathrm{d}\, x^j \\
&= \frac{\partial y^\sigma}{\partial \bar y^\nu}\,\mathrm{d}\,\bar y^\nu + \frac{\partial y^\sigma}{\partial \bar x^i}\,\mathrm{d}\,\bar x^i - \bar{\mathrm{D}}_i y^\sigma \frac{\partial \bar x^i}{\partial x^j}\frac{\partial x^j}{\partial \bar x^k}\,\mathrm{d}\,\bar x^k,
\end{aligned}$$

where the index k is sumed over 1 to n,

$$\begin{aligned}
&= \frac{\partial y^\sigma}{\partial \bar y^\nu}\,\mathrm{d}\,\bar y^\nu + \frac{\partial y^\sigma}{\partial \bar x^i}\,\mathrm{d}\,\bar x^i - \left(\frac{\partial y^\sigma}{\partial \bar x^i} + \frac{\partial y^\sigma}{\partial \bar y^\nu}\bar y^\nu_i\right)\mathrm{d}\,\bar x^i \\
&= \frac{\partial y^\sigma}{\partial \bar y^\nu}\left(\mathrm{d}\,\bar y^\nu - \bar y^\nu_i \,\mathrm{d}\,\bar x^i\right) \\
&= \frac{\partial y^\sigma}{\partial \bar y^\nu}\bar\omega^\nu.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned} I(\rho) &= \tfrac{1}{k}\omega^\sigma \wedge C_\sigma \\ &= \tfrac{1}{k}\frac{\partial y^\sigma}{\partial \bar{y}^\nu}\bar{\omega}^\nu \wedge \frac{\partial \bar{y}^\lambda}{\partial y^\sigma}\bar{C}_\lambda, \end{aligned}$$

where the summation over λ extends from 1 to m,

$$= \tfrac{1}{k}\bar{\omega}^\nu \wedge \bar{C}_\nu.$$

Clearly, we have shown that $I(\rho) \in \Omega^{2r+1}_{n+k}W$ is a well-defined differential $(n+k)$-form on W.

Remark 7. Let Ξ be a vertical vector field on Y, J^rY its prolongation. In local coordinates we write $\Xi = \Xi^\sigma \partial_\sigma$ and $J^r\Xi = \Xi^\sigma_J \partial^J_\sigma$, where $\Xi^\sigma_J = \mathrm{D}_J\,\Xi^\sigma$. Recall that $\omega^\sigma_J(\partial^I_\nu) = \delta^\sigma_\nu \delta^I_J$ and $\mathrm{D}_J\,\omega^\sigma = \omega^\sigma_J$. Clearly, we can exploit the duality between vertical vector fields and contact 1-forms and use it to obtain the analogy of Theorem IV.2 for the contraction Euler mapping I. In particular, let α be a contact form on J^rY. If $J^r\Xi \lrcorner\, \alpha = 0$, we also have $\alpha = 0$.

Lemma IV.3. *For any $\eta \in \Omega^r_{n+k}W$, it holds that $I \circ \mathrm{p}_k \circ \mathrm{d} \circ \mathrm{p}_k\,\eta = 0$.*

Proof. The form $I(\mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_k\,\eta)$ for $\eta \in \Omega^r_{n+k}W$, $k \geq 1$ is constructed using (IV.18) in which $C_\sigma \in \mathrm{p}_k\,\Omega^{2r+1}_{n+k}W$ are coefficients of the contraction Euler operator corresponding to the formal differential operator $C_{\mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_k\,\eta} = J^{r+2}\Xi \lrcorner\, \mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_k\,\eta$ (see (IV.5).

$$C_{\mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_k\,\eta} = J^{r+2}\Xi \lrcorner\, \mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_k\,\eta = J^{r+2} \lrcorner \left[(-1)^{n+k}\,\mathrm{D}_j(\mathrm{p}_k\,\rho) \wedge \mathrm{d}\,x^j\right].$$

On the other hand using (III.37) we have

$$\begin{aligned} C_{\mathrm{p}_k\,\mathrm{d}\,\mathrm{p}_k\,\eta} &= -\,\mathrm{p}_{k-1}\,\mathrm{d}(J^{r+1} \lrcorner\, \mathrm{p}_k\,\eta) = -\,\mathrm{p}_{k-1}\,\mathrm{d}\,\mathrm{p}_{k-1}(J^{r+1}\Xi \lrcorner\, \mathrm{p}_k\,\eta) \\ &= -(-1)^{n+k-1}\,\mathrm{D}_j(J^{r+1}\Xi \lrcorner\, \mathrm{p}_k\,\eta) \wedge \mathrm{d}\,x^j = (-1)^{n+k}(\mathrm{D}_j\,C_{\mathrm{p}_k\,\eta}) \wedge \mathrm{d}\,x^j. \end{aligned}$$

Thus,

$$J^{r+2} \lrcorner (\mathrm{D}_j(\mathrm{p}_k\,\eta) \wedge \mathrm{d}\,x^j) = \mathrm{D}_i(\underbrace{J^{r+1}\Xi \lrcorner\, \mathrm{p}_k\,\eta}_{C_{\mathrm{p}_k\,\eta}}) \wedge \mathrm{d}\,x^i.$$

Using (IV.9) we have

$$\sum_{|J|=0}^{r+2} \mathrm{D}_J \left\{ \Xi^\sigma C^J_\sigma \left[(-1)^{n+k} \, \mathrm{D}_j(\mathrm{p}_k \, \eta) \wedge \mathrm{d}\, x^j \right] \right\} =$$

$$= (-1)^{n+k} \, \mathrm{D}_i \sum_{|I|=0}^{r+1} \mathrm{D}_I \left[\Xi^\sigma C^I_\sigma (\mathrm{p}_k \, \eta \wedge \mathrm{d}\, x^i) \right] =$$

$$= (-1)^{n+k} \sum_{|J|=1}^{r+2} \mathrm{D}_J \left[\Xi^\sigma C^{(I}_\sigma (\mathrm{p}_k \, \eta \wedge \mathrm{d}\, x^{i)}) \right],$$

where $J = Ii$. As the coefficients C^I_σ are unique, we have

$$C^J_\sigma [\mathrm{D}_j(\mathrm{p}_k \, \eta) \wedge \mathrm{d}\, x^j] = C^{(I}_\sigma (\mathrm{p}_k \, \eta \wedge \mathrm{d}\, x^{i)})$$

for $J = Ii$ and in particular

$$C_\sigma [\mathrm{D}_j(\mathrm{p}_k \, \eta) \wedge \mathrm{d}\, x^j] = 0$$

and consequently $I(\mathrm{p}_k \, \mathrm{d} \, \mathrm{p}_k \, \eta) = 0$. □

We shall further investigate the properties of the contraction Euler mapping in the following chapter.

Chapter V

The Variational Sequence

We follow [34] and describe a suitable subsequence of the de Rham sequence of sheaves on the r-jet prolongation of a fibered manifold and define the variational sequence as the corresponding quotient sequence. We show how the variational sequence relates to the classical objects known from the calculus of variations such as Lagrangians, Euler-Lagrange expressions etc. Finally, we comment on how the variational sequence determines the local and global properties of objects in the calculus of variations.

1. Elementals of Sheaf Theory

1.1. Sheaves of differential forms. Let us start with a brief excursion into the theory of sheaves. We shall discuss sheaves of Abelian groups; analogously we would work with sheaves of groups, rings and modules. Let Y be a topological space and suppose that the following conditions are satisfied

(i) there exists an Abelian group $\mathscr{F}(U)$ for each open subset $U \subset Y$, and $\mathscr{F}(\emptyset) = \{0\}$,

(ii) there exists a morphism $\iota_{U,V} \colon \mathscr{F}(V) \to \mathscr{F}(U)$ for each pair of open sets $U \subset V \subset Y$, such that $\iota_{U,U} = \{1\}$ (the identity in the Abelian group $\mathscr{F}(U)$) and $\iota_{U,W} = \iota_{U,V} \circ \iota_{V,W}$ for all $U \subset V \subset W \subset Y$.

We call $\mathscr{F}$, consisting of the family $\{\mathscr{F}(U)\}$ and the family of mappings $\{\iota_{U,V}\}$, a *presheaf* of Abelian groups on Y. For brevity we write

$$\mathscr{F}(V) \ni \gamma \to \iota_{UV}(\gamma) \in \mathscr{F}(U) = \gamma|_U \, ,$$

whenever $U \subset V \subset Y$ and call it the *restriction* of γ to U. A *morphism* $\varphi \colon \mathscr{F} \to \mathscr{G}$ between two presheaves $\mathscr{F}$ and $\mathscr{G}$ on Y is a family $\{\varphi(U)\}$ of Abelian group homomorphisms $\varphi(U) \colon \mathscr{F}(U) \to \mathscr{G}(U)$ satisfying the condition $\iota_{U,V} \circ \varphi(V) = \varphi(U) \circ \iota_{U,V}$ for each $U \subset V \subset Y$.

A presheaf $\mathscr{F}$ is called a sheaf if it satisfies the following condition: If $U \subset Y$ is an open set and $\{U_\kappa\}_{\kappa \in K}$ is an open cover of U, and if for each $\kappa \in K$ an element $\gamma_\kappa \in \mathscr{F}(U_\kappa)$ is given such that $\gamma_\kappa|_{U_\kappa \cap U_\lambda} = \gamma_\lambda|_{U_\kappa \cap U_\lambda}$ for all $\kappa, \lambda \in K$, then there exists a unique $\gamma \in \mathscr{F}(U)$ such that $\gamma|_{U_\kappa} = \gamma_\kappa$ for all $\kappa \in K$. Let $\mathscr{F}$ be a presheaf, $y \in Y$ be a point of the topological space Y and U, V be neighborhoods of the point y. Let $\gamma \in \mathscr{F}(U)$ and $\delta \in \mathscr{F}(V)$. We define an equivalence relation by $\gamma \sim \delta$ if and only if there exists a neighborhood W of the point y, such that $\gamma|_W = \delta|_W$. The equivalence classes are called *germs* of γ at the point y, and the set of all equivalence classes the *stalk* of $\mathscr{F}$ over y denoted by $\mathscr{F}_y$. Morphisms of presheaves induce morphisms of stalks.

We shall topologize the direct sum $\mathscr{F}' = \coprod_{y \in Y} \mathscr{F}_y$. We take all $U \subset Y$, open subsets of Y and all $\gamma \in \mathscr{F}(U)$. As the base for open sets we take the sets of germs of γ at all points y of the open set U. In other words, we have chosen the topology in such a way that the mapping $\pi \colon \mathscr{F}' \to Y$ that maps the germs in $\mathscr{F}_y$ to the point y is continuous, and each $\pi^{-1}(y) = \mathscr{F}_y$ is an Abelian group. Moreover, π is a local homeomorphism and the group operations

$$\mathscr{F}' \times_Y \mathscr{F}' = \{(\gamma, \delta) | \pi(\gamma) = \pi(\delta)\} \ni (\gamma, \delta) \to \gamma + \delta \in \mathscr{F}'$$
$$\mathscr{F}' \ni \gamma \to -\gamma \in \mathscr{F}'$$

on $\pi^{-1}(y)$ are continuous. The topological space $\mathscr{F}'$ is called a *sheaf space*. Let $A \subset Y$ be a topological subspace. A continuous mapping $\gamma \colon A \to \mathscr{F}'$, such that $\pi \circ \gamma = \{1\}_A$ is called a *section* over A. The set of sections over A, denoted by $\Gamma(A, \mathscr{F}')$ is obviously an Abelian group. We associate the Abelian group $\Gamma(U, \mathscr{F}')$ with each open subset $U \subset Y$ and define $\iota_{U,V}$ by restrictions of sections $\{\iota_{U,V}(\gamma) = \gamma|_U\}$. We obtain a sheaf $\mathscr{F}''$, called the *sheaf associated with the presheaf* $\mathscr{F}$. If $\mathscr{F}$ is a sheaf, we can prove that $\mathscr{F}''$ is isomorphic to $\mathscr{F}$. Conversely, if we start from a sheaf space $\mathscr{F}'$, then construct the sheaf $\mathscr{F}''$ and finally the sheaf space $\mathscr{F}'''$, then $\mathscr{F}'''$ is canonically isomorphic to $\mathscr{F}'$. Thus, we can identify a sheaf and its corresponding sheaf space.

Example 4. Let us name some important examples of sheaves, which will be used in the following.

- **The trivial sheaf.** We take the trivial group $\{1\}$ and a topological space Y. The Cartesian product $Y \times \{1\}$ is called the trivial sheaf.
- **The constant sheaf.** Let G be an Abelian group (or some other algebraic structure), considered with discrete topology and let Y be a topological space. The Cartesian product $Y \times G$ is called the constant sheaf.
- **The sheaf of continuous real valued functions.** Let Y be a topological space and $\mathbb{R}$ the topological group of real numbers. We obtain a sheaf $\mathscr{F}$ on Y by putting $\mathscr{F}(U) = \{\gamma \colon U \to \mathbb{R}\}$, where the mapping γ is continuous and the restrictions are defined naturally (i.e. as restrictions of the domains of the mappings γ). This sheaf has the structure of a sheaf of commutative rings with unity.
- **Sheaf of germs of functions of class** C^r, where $r = 1, 2, \dots, \infty$, on a smooth manifold. This sheaf is defined analogously as the sheaf of continuous functions. We require the functions to be of class C^r.
- **The sheaf of germs of sections of a vector bundle.** Given a vector bundle $\pi \colon E \to Y$, we define a sheaf on Y by $\mathscr{F}(U) = \Gamma(E, U)$ (the module of sections over U with restrictions defined naturally. The stalk at the point $y \in Y$ consists of germs at y of sections of the vector bundle E.
- **The sheaf of germs of differentiable q-forms.** For the vector bundle, mentioned in the previous example, we take the q-th exterior power of the cotangent bundle of the differentiability class C^r. This is a sheaf with the structure of a module over the ring of functions of the class C^r.

1.2. Sequences of Abelian groups. A family $A^* = \{(A^i, d^i)\}$ of Abelian groups A^i and morphisms $d^i \colon A^i \to A^{i+1}$ indexed by integers $i \in \mathbb{Z}$ is called a *sequence of Abelian groups*. A sequence of Abelian groups may begin and end by an infinite string of trivial groups and the corresponding trivial morphisms. In such a case, the Abelian sequence is said to be *finite* and may be written as

$$\{0\} \longrightarrow A^r \xrightarrow{d^r} A^{r+1} \xrightarrow{d^{r+1}} \cdots \xrightarrow{d^{s-2}} A^{s-1} \xrightarrow{d^{s-1}} A^s \longrightarrow \{0\}.$$

A sequence of Abelian groups is said to be *exact at a term* A^q, if $\ker d^q = \operatorname{im} d^{q-1}$. An Abelian sequence A^* is called *exact* if it is exact at all terms

A^q, $q \in \mathbb{Z}$. An exact sequence of the form

$$\{0\} \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \{0\} \tag{V.1}$$

is called a *short exact sequence*. Next, we list various properties of short exact sequences.

Lemma V.1. *(a) The sequence (V.1) is exact at C if and only if g is surjective.*

(b) The sequence (V.1) is exact at A if and only if f is injective.

(c) The sequence of Abelian groups

$$\{0\} \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} B/A \longrightarrow \{0\},$$

where $A \subset B$ is a subgroup, $\iota\colon A \to B$ the inclusion, B/A the quotient group and $\pi B \to B/A$ the quotient projection, is a short exact sequence.

(d) Assume we are given the following scheme

$$\begin{tikzcd}
\{0\} \arrow[r] & A \arrow[r, "f"] \arrow[d, "\varphi"] & B \arrow[r, "g"] \arrow[d, "\chi"] & C \arrow[r] & \{0\} \\
\{0\} \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & \{0\},
\end{tikzcd}$$

where the horizontal sequences are short exact sequences of Abelian groups, φ and χ are group morphisms and the first square commutes, i.e. $\chi \circ f = f' \circ \varphi$. Then there exists a unique group morphism $\psi\colon C \to C'$ such that the second square of the scheme

$$\begin{tikzcd}
\{0\} \arrow[r] & A \arrow[r, "f"] \arrow[d, "\varphi"] & B \arrow[r, "g"] \arrow[d, "\chi"] & C \arrow[r] \arrow[d, "\psi"] & \{0\} \\
\{0\} \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & \{0\}
\end{tikzcd}$$

commutes.

(e) Let

$$\{0\} \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \{0\}$$

be an exact sequence of Abelian groups, $\pi\colon B \to B/f(A)$, the quotient projection. There exists a unique group isomorphism

$$\psi\colon C \to B/f(A)$$

such that the diagram

$$\begin{tikzcd}
\{0\} \arrow[r] & A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"] & B \arrow[r, "g"] \arrow[d, "\mathrm{id}_B"] & C \arrow[r] \arrow[d, "\psi"] & \{0\} \\
\{0\} \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "\pi"] & B/f(A) \arrow[r] & \{0\}
\end{tikzcd}$$

commutes.

Proof. All proofs are simple consequences of definitions and can be found in [42]. □

A sequence of Abelian groups $A^* = \{(A^i, d^i)\}_{i\in\mathbb{Z}}$ is called a *complex* if $d^{i+1} \circ d^i = 0$ for all $i \in \mathbb{Z}$. The family of Abelian group morphisms $d^* = \{d^i\}_{i\in\mathbb{Z}}$ is called the *differential* of the complex A^*. The quotient group

$$H^q A^* = \ker d^q / \operatorname{im} d^{q-1}$$

is called the q-th *cohomology group* of the complex A^*, the group elements are called the q-th *cohomology classes* of A^*. An exact sequence of Abelian groups forms a complex with *trivial* cohomology groups. If A is an Abelian group, then any exact sequence of the form

$$\{0\} \longrightarrow A \xrightarrow{\epsilon} B^0 \xrightarrow{d^0} B^1 \xrightarrow{d^1} B^2 \xrightarrow{d^2} \cdots \tag{V.2}$$

is called a *resolution* of A. The resolution (V.2) defines a complex $B^* = \{(B^i, d^i)\}$, i.e.,

$$\{0\} \longrightarrow B^0 \xrightarrow{d^0} B^1 \xrightarrow{d^1} B^2 \xrightarrow{d^2} \cdots \tag{V.3}$$

such that $H^0B^* = A$ and $H^iB^* = \{0\}$ for $i \in \mathbb{N}$.

Note that all previously defined concepts can be applied to sheaves over the same base and to sheaf morphisms. We shall also define some additional structures. To this end, let $\mathscr{F}^*$, usually written as

$$\cdots \longrightarrow \mathscr{F}^{q-1} \xrightarrow{\delta^{q-1}} \mathscr{F}^q \xrightarrow{\delta^q} \mathscr{F}^{q+1} \xrightarrow{\delta^{q+1}} \cdots$$

be a *sequence of sheaves*. We shall consider the sequence of stalks of sheaves $\mathscr{F}^q$ over the point y and correspondingly restrict the sheaf morphisms δ^q to these stalks. We get a sequence of Abelian groups

$$\cdots \longrightarrow \mathscr{F}_y^{q-1} \xrightarrow{\delta_y^{q-1}} \mathscr{F}_y^q \xrightarrow{\delta_y^q} \mathscr{F}_y^{q+1} \xrightarrow{\delta_y^{q+1}} \cdots .$$

This sequence is said to be the *restriction* of the sequence $\mathscr{F}^*$ to the point $y \in Y$. The sequence of sheaves $\mathscr{F}^*$ is said to be *exact at $\mathscr{F}^q$ over the point y*, if its restriction to y is exact at $\mathscr{F}_y^q$ and *exact at $\mathscr{F}^q$* if the previous assumption holds for all $y \in Y$. Finally, we say that $\mathscr{F}^*$ is an *exact* sequence if it is exact at $\mathscr{F}^q$ for all $q \in \mathbb{Z}$. A sequence of sheaves $\mathscr{F}^* = (\mathscr{F}^q, \delta^q)$ fulfilling the condition $\delta^{q+1} \circ \delta^q = 0$ for all $q \in \mathbb{Z}$ is called a *differential sequence*. An exact sequence is obviously differential. One also defines a *resolution* of $\mathscr{F}$ as the exact sequence of the form

$$0 \longrightarrow \mathscr{F} \xrightarrow{\epsilon} \mathscr{F}^0 \xrightarrow{\delta^0} \mathscr{F}^1 \xrightarrow{\delta^1} \mathscr{F}^2 \xrightarrow{\delta^2} \cdots .$$

We also introduce short exact sequences of sheaves analogously. If a sequence is exact, then its restriction to a topological subspace $Y' \subset Y$ is also exact.

A sequence of sheaves

$$\text{(V.4)} \qquad \cdots \longrightarrow \mathscr{F}^{q-1} \xrightarrow{\delta^{q-1}} \mathscr{F}^q \xrightarrow{\delta^q} \mathscr{F}^{q+1} \xrightarrow{\delta^{q+1}} \cdots$$

over the topological space Y defines a sequence of Abelian groups of continuous sections $\mathscr{F}^q(U)$ for every open subset $U \subset Y$, together with Abelian group morphisms $\delta^q_U \colon \mathscr{F}^q(U) \to \mathscr{F}^{q+1}(U)$. The sequence of continuous sections given by

$$\text{(V.5)} \qquad \cdots \longrightarrow \mathscr{F}^{q-1}U \xrightarrow{\delta_U^{q-1}} \mathscr{F}^q U \xrightarrow{\delta_U^q} \mathscr{F}^{q+1}U \xrightarrow{\delta_U^{q+1}} \cdots$$

is said to be *induced* by the sequence of sheaves (V.4). In particular, for $U = Y$, this sequence is called the *induced sequence of global sections*. Exactness of the sequence (V.4) by no means implies the exactness of the sequence (V.5).

Assume that we are given a sheaf $\mathscr{F}$ over a topological space Y. We wish to introduce a resolution of the sheaf $\mathscr{F}$

$$0 \longrightarrow \mathscr{F} \xrightarrow{\epsilon} \mathscr{C}^0(\mathscr{F}) \xrightarrow{\epsilon^0} \mathscr{C}^1(\mathscr{F}) \xrightarrow{\epsilon^1} \mathscr{C}^2(\mathscr{F}) \xrightarrow{\epsilon^2} \cdots ,$$

where the sheaves $\mathscr{C}^q(\mathscr{F})$ would be constructed from $\mathscr{F}$ by means of short exact sequences. This is indeed possible, though we shall consider presheaves of sections, which are not necessarily continuous. We define $\mathscr{C}^0(\mathscr{F}) = \operatorname{germ}\sec\mathscr{F}$, where $\sec\mathscr{F}$ denotes the presheaf of (not necessarilly continuous) sections of $\mathscr{F}$. Since $\mathscr{F}$ is canonically isomorphic with $\operatorname{germ}\sec^{\mathrm{cont}}\mathscr{F} \subset \operatorname{germ}\sec\mathscr{F}$, where $\sec^{\mathrm{cont}}\mathscr{F}$ is the presheaf of continuous sections of $\mathscr{F}$. We denote the corresponding canonical morphism induced by this inclusion by $\epsilon \colon \mathscr{F} \to \mathscr{C}^0(\mathscr{F})$. The mapping ϵ is obviously injective. Thus we can construct the quotient sheaf $\bar{\mathscr{C}}^1(\mathscr{F}) = \mathscr{C}^0(\mathscr{F}) / \operatorname{im}\epsilon$. We obtain the short exact sequence of sheaves

$$0 \longrightarrow \mathscr{F} \xrightarrow{\epsilon} \mathscr{C}^0(\mathscr{F}) \xrightarrow{\pi} \bar{\mathscr{C}}^1(\mathscr{F}) \longrightarrow 0,$$

where π is the quotient projection. We define

$$\mathscr{C}^1(\mathscr{F}) = \mathscr{C}^0(\bar{\mathscr{C}}^1(\mathscr{F})) = \operatorname{germ}\sec\bar{\mathscr{C}}^1(\mathscr{F}),$$

and

$$\bar{\mathscr{C}^q}(\mathscr{F}) = \mathscr{C}^{q-1}(\mathscr{F})/\bar{\mathscr{C}^{q-1}}(\mathscr{F}),$$
$$\mathscr{C}^q(\mathscr{F}) = \mathscr{C}^0(\bar{\mathscr{C}^{q-1}}(\mathscr{F})) = \operatorname{germ}\sec\bar{\mathscr{C}^{q-1}}(\mathscr{F}),$$

for $q \geq 2$. We obtain short exact sequences

$$0 \longrightarrow \bar{\mathscr{C}^q}(\mathscr{F}) \longrightarrow \mathscr{C}^{q+1}(\mathscr{F}) \longrightarrow \bar{\mathscr{C}^{q+1}}(\mathscr{F}) \longrightarrow 0,$$

for $q \geq 1$. Combining these short exact sequences we get the scheme

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[dr] & & 0 \arrow[dr] & & 0 \arrow[dr] & & \cdots \\
 & & \bar{\mathscr{C}^1}(\mathscr{F}) \arrow[ur] \arrow[dr] & & \bar{\mathscr{C}^2}(\mathscr{F}) \arrow[ur] \arrow[dr] & & \bar{\mathscr{C}^q}(\mathscr{F}) \arrow[ur] \arrow[dr] & \\
0 \to \mathscr{F} \to & \mathscr{C}^0(\mathscr{F}) \arrow[ur] \arrow[rr] & & \mathscr{C}^1(\mathscr{F}) \arrow[ur] \arrow[rr] & & \mathscr{C}^2(\mathscr{F}) \arrow[ur] \arrow[rr] & & \cdots,
\end{tikzcd}$$

whose bottom row defines the canonical resolution of the sheaf $\mathscr{F}$. We shall consider global continuous sections of every term of the canonical resolution of the sheaf $\mathscr{F}$ and obtain the sequence of Abelian groups

$$0 \longrightarrow \mathscr{F}Y \xrightarrow{\epsilon} \mathscr{C}^0(\mathscr{F})Y \xrightarrow{\epsilon^0} \mathscr{C}^1(\mathscr{F})Y \xrightarrow{\epsilon^1} \mathscr{C}^2(\mathscr{F})Y \xrightarrow{\epsilon^2} \cdots.$$

We denote by $\mathscr{C}^*(\mathscr{F})Y$ the complex

$$0 \longrightarrow \mathscr{C}^0(\mathscr{F})Y \xrightarrow{\epsilon^0} \mathscr{C}^1(\mathscr{F})Y \xrightarrow{\epsilon^1} \mathscr{C}^2(\mathscr{F})Y \xrightarrow{\epsilon^2} \cdots.$$

For every $q \geq 0$ we define $H^q(Y, \mathscr{F}) = H^q(\mathscr{C}^*(\mathscr{F})Y)$. The Abelian group $H^q(Y, \mathscr{F})$ is called the q-th *cohomology group of Y with coefficients in the sheaf $\mathscr{F}$*.

A sheaf $\mathscr{F}$ is called *soft* if every continuous section of $\mathscr{F}$ defined on a closed subset of Y can be prolonged to a global continuous section. If Y is paracompact and Y' is a closed subspace of Y, then any closed subset of Y' is closed in Y. Therefore, if $\mathscr{F}$ is a soft sheaf, then so is any restriction of $\mathscr{F}$.

Theorem V.2. *Let Y be a paracompact Hausdorff space and*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\chi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \longrightarrow 0 \tag{V.6}$$

a short exact sequence. If the sheaf $\mathscr{F}$ is soft, then the sequence of Abelian groups

$$0 \longrightarrow \mathscr{F}Y \xrightarrow{\chi_Y} \mathscr{G}Y \xrightarrow{\psi_Y} \mathscr{H}Y \longrightarrow 0 \tag{V.7}$$

is exact.

Proof. The proof can be found [42]. □

In particular, if the sheaves $\mathscr{F}$ and $\mathscr{G}$ in the short exact sequence (V.6) are soft, then the sheaf $\mathscr{H}$ is also soft. Also, the sheaves $\mathscr{C}^q\mathscr{F}$, $q \geq 0$, in the canonical resolution of the sheaf $\mathscr{F}$ are soft. If Y is a paracompact Hausdorff space and

$$0 \longrightarrow \mathscr{F}^0 \xrightarrow{\delta^0} \mathscr{F}^1 \xrightarrow{\delta^1} \mathscr{F}^2 \xrightarrow{\delta^2} \cdots \tag{V.8}$$

is an exact sequence of sheaves over Y, then all of the sheaves $\mathscr{F}^q$, $q \geq 0$, are soft, and the induced sequence of Abelian groups

$$0 \longrightarrow \mathscr{F}^0 Y \xrightarrow{\delta^0_Y} \mathscr{F}^1 Y \xrightarrow{\delta^1_Y} \mathscr{F}^2 Y \xrightarrow{\delta^2_Y} \cdots \tag{V.9}$$

is exact. If $\mathscr{F}$ is a soft sheaf over a paracompact Hausdorff space Y, then $H^q(X, \mathscr{F}) = 0$ for all $q \geq 1$.

The set $\operatorname{supp}\delta = \operatorname{cl}\left\{y \in Y | \delta|_{\mathscr{F}_y} \neq 0\right\}$ is called the support of the morphism $\delta\colon \mathscr{F} \to \mathscr{F}$. By a *sheaf partition of unity* on $\mathscr{F}$, subordinate to the open covering $\{U_\iota\}_\iota$ of Y, we mean the family $\{\eta_\iota\}_\iota$ of sheaf morphisms $\eta_\iota\colon \mathscr{F} \to \mathscr{F}$ such that $\operatorname{supp}\eta_\iota \subset U_\iota$ for all ι and if each point $y \in Y$ has a neighborhood U such that $\eta_\iota|_U \neq 0$ for only a finite set of indices ι, and for all $\gamma \in \mathscr{F}$

$$\sum_\iota \eta_\iota(\gamma) = \gamma.$$

The previous sum is in fact finite. A sheaf $\mathscr{F}$ is said to be *fine* if to every locally finite open covering $\{U_\iota\}$ of Y, there exists a sheaf partition of unity of $\mathscr{F}$ subordinate to $\{U_\iota\}$. If a sheaf $\mathscr{F}$ is fine, then it is soft.

A sheaf $\mathscr{F}$ over Y is said to be *acyclic*, if $H^q(Y, \mathscr{F}) = 0$ for $q \geq 1$. A resolution of $\mathscr{F}$

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}^0 \longrightarrow \mathscr{F}^1 \longrightarrow \mathscr{F}^2 \longrightarrow \cdots$$

is said to be *acyclic*, if each of the sheaves $\mathscr{F}^i$, $i \geq 0$ is acyclic. In particular, if $\mathscr{F}$ is a sheaf over a paracompact Hausdorff space Y, then if $\mathscr{F}$ is soft, it is acyclic and the canonical resolution is acyclic.

Theorem V.3 (Abstract de Rham Theorem)**.** *If $\mathscr{F}$ is a sheaf over a paracompact Hausdorff space Y, and if*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\epsilon} \mathscr{F}^0 \longrightarrow \mathscr{F}^1 \longrightarrow \mathscr{F}^2 \longrightarrow \cdots$$

is an acyclic resolution of $\mathscr{F}$, then for all $q \geq 0$ the cohomology groups $H^q(Y, \mathscr{F})$ and $H^q(\mathscr{F}^* Y)$ are isomorphic.

Proof. The proof can be found in [42]. □

Example 5 (The de Rham sequence)**.** Let Y be a smooth N-dimensional manifold, $\mathbb{R}_Y$ the constant sheaf over Y, Ω_q the presheaf of smooth q-forms on Y, $U \subset Y$ open. The exterior derivative of differential forms defines the sequence of Abelian groups

$$0 \longrightarrow \mathbb{R}_Y U \xrightarrow{\mathrm{d}} \Omega_0 U \xrightarrow{\mathrm{d}} \Omega_1 U \xrightarrow{\mathrm{d}} \Omega_2 U \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega_N U \xrightarrow{\mathrm{d}} 0. \tag{V.10}$$

If the conditions of the Volterra-Poincaré Lemma are satisfied, i.e. U is open and star-shaped, then the sequence (V.10) is exact. The sequence need not be exact otherwise. We commonly denote the sheaves germ Ω_q by the same letter as the corresponding presheaves. Each of the sheaves Ω_q in the de Rham sequence is fine hence soft hence acyclic. The de Rham sequence is an acyclic resolution of the constant sheaf $\mathbb{R}_Y$.

2. The variational sequence

We shall assume that the manifold Y from the previous Example 5 has the structure of a fibered manifold $\pi\colon Y \to X$. To each open set $W \subset Y$ we assign the ring of smooth functions $\Omega_0^r W$ defined on the open set $W^r = (\pi^{r,0})^{-1}(W)$. To each two open sets $V \subset W \subset Y$, we assign the canonical inclusion $\iota_{V,W}\colon V^r \to W^r$. The collections $\{\Omega_0^r W\}$ and $\{\iota^*_{V,W}\}$, where $\iota^*_{V,W}$ is the pull-back mapping (i.e. restriction), define a sheaf of commutative rings with unity over Y, denoted by Ω_0^r. Let $q > 0$ be an integer. We assign to each open set $W \subset Y$ the module of smooth differential q-forms $\Omega_q^r W$ defined on the open set $W^r = (\pi^{r,0})^{-1}(W)$. The $\{\Omega_q^r W\}$ and $\{\iota^*_{V,W}\}$, where $\iota^*_{V,W}$ is the pull-back mapping (i.e. restriction), define a sheaf of Ω_0^r-modules over Y denoted by Ω_q^r. From the construction above we gather that, for every $q \geq 0$, the sheaf Ω_q^r is the direct image of the sheaf of q-forms over $J^r Y$ by the pullback of the canonical jet projection $\pi^{r,0}$.

2.1. The Poincaré Lemma for prolongations of fibered manifolds. We consider an open set $U \subset \mathbb{R}^n$, and an open ball $V \subset \mathbb{R}^m$ centered at the point $0 \in \mathbb{R}^m$. Let $W = U \times V$ and denote by $\tau\colon W \to U$ the canonical projection on the first factor. Let (x^i, y^σ) be the canonical coordinates on W. We define the mapping

$$\chi_r\colon [0,1] \times J^r W \ni (t, (x^i, y^\sigma_J)) \to \chi_r(t, (x^i, y^\sigma_J)) = (x^i, t y^\sigma_J) \in J^r W. \tag{V.11}$$

The mapping χ_r has the following more or less obvious properties

$$\chi_r(0, (x^i, y^\sigma_J)) = (x^i, 0_J), \tag{V.12}$$

$$\chi_r(1, (x^i, y^\sigma_J)) = (x^i, y^\sigma_J), \tag{V.13}$$

where $0 \leq |J| \leq r$. For pull-backs of differential forms by χ_r it holds

$$\chi_r^* \,\mathrm{d}\, x^i = \mathrm{d}\, x^i, \tag{V.14}$$

$$\chi_r^* \,\mathrm{d}\, y^\sigma_J = t \,\mathrm{d}\, y^\sigma_J + y^\sigma_J \,\mathrm{d}\, t, \tag{V.15}$$

where $0 \leq |J| \leq r$,

$$\chi_r^* \omega_J^\sigma = t\omega_J^\sigma + y_J^\sigma \,\mathrm{d}\, t, \tag{V.16}$$

where $0 \leq |J| \leq r-1$. Obviously, for any q-form $\rho \in \Omega_q^r W$, $q \geq 1$, the pull-back $\chi_r^* \rho$ can be uniquely written as

$$\chi_r^* \rho = \mathrm{d}\, t \wedge \rho_0(t) + \rho_1(t), \tag{V.17}$$

the forms $\rho_0(t)$ and $\rho_1(t)$ not containing the factor $\mathrm{d}\, t$. We set

$$A\rho(\xi_1, \dots, \xi_{q-1}) = \int_0^1 \rho_0(t)(\xi_1, \dots, \xi_{q-1}) \,\mathrm{d}\, t. \tag{V.18}$$

for arbitrary vector fields $\xi_1, \dots \xi_{q-1}$ on $J^r W$. The operator A is called the *contact homotopy operator*. Denote by

$$\zeta_0 \colon U \ni (x^i) \to (x^i, 0_J) \in J^r W$$

the canonical zero section.

2.2. The factorisation subsequence. The mappings $\mathrm{p}_q \colon \Omega_q^r W \to \Omega_q^{r+1} W$ defined in (III.2) induce the appropriate sheaf morphisms $\mathrm{p}_q \colon \Omega_q^r \to \Omega_q^{r+1}$. Since this can not lead to any misunderstandings, we shall use the same notation. We denote $\Omega_{q,\mathrm{c}}^r = \ker \mathrm{h}$ for $0 \leq q \leq n$ and similarly $\Omega_{q,\mathrm{c}}^r = \ker \mathrm{p}_{q-n}$ for $n+1 \leq q$. These sheaves are locally described by Theorems III.4 and III.5. The exterior derivative d induces the sheaf morphism $\mathrm{d}_q \colon \Omega_q^r \to \Omega_{q+1}^r$. The image of the sheaf $\Omega_{q,\mathrm{c}}^r$ by the mapping d_q will be denoted by $\mathrm{d}\, \Omega_{q,\mathrm{c}}^r$. We set

$$\begin{aligned} \Theta_1^r &= \Omega_{1,\mathrm{c}}^r \\ \Theta_q^r &= \mathrm{d}\, \Omega_{q-1,\mathrm{c}}^r + \Omega_{q,\mathrm{c}}^r \text{ for } q \geq 2 \end{aligned} \tag{V.19}$$

The sheaves Θ_q^r, with

$$q > P = m\tbinom{n+r-1}{n} + 2n - 1$$

are trivial. We shall investigate the sequence of sheaves of C^∞-modules of differential forms given as

$$\{0\} \longrightarrow \Theta_1^r \xrightarrow{\mathrm{d}_1} \Theta_2^r \xrightarrow{\mathrm{d}_2} \cdots \xrightarrow{\mathrm{d}_{P-1}} \Theta_P^r \xrightarrow{\mathrm{d}_P} \{0\}. \tag{V.20}$$

Theorem V.4. *Let $\rho \in \Omega_q^r W$, $q \geq 1$, be a differential form. It holds*

$$\rho = A \,\mathrm{d}\, \rho + \mathrm{d}\, A\rho + (\pi^r)^* \zeta_0^* \rho, \tag{V.21}$$

and

$$\mathrm{p}_{k-1}\, A\rho = A\, \mathrm{p}_k\, \rho, \quad \textit{for } 1 \leq k \leq q. \tag{V.22}$$

Proof. The proof of the first claim can be verified by writing the equation down in a fibered chart. The second claim is evident from the construction of the operator A. $\square$

Lemma V.5. *If a q-form $\rho \in \Omega^r_q W$ belongs to the ideal of the exterior algebra of forms generated by the contact forms ω^σ_J, $0 \le |J| \le r-1$, and $\mathrm{d}\,\omega^\sigma_I$, $|I| = r-1$, see (2.1), then*

$$\rho = A\,\mathrm{d}\,\rho + \mathrm{d}\,A\rho. \tag{V.23}$$

Proof. The proof is the direct consequence of the previous Theorem V.4. $\square$

Lemma V.6. *The sequence of sheaves (V.20) of Abelian groups is exact.*

Proof. We shall first prove exactness of the sequence (V.20) at Θ^r_1. We pick an element $\alpha \in \Theta^r_1$. In a fibered chart, the element α is expressed as $\alpha = A^J_\sigma \omega^\sigma_J$, $0 \le |J| \le r-1$, see Theorem III.4. Then we obtain

$$\mathrm{p}_1\,\mathrm{d}\,\alpha = -(\mathrm{D}_j\,A^J_\sigma)\omega^\sigma_J \wedge \mathrm{d}\,x^j - A^J_\sigma \omega^\sigma_{Jj} \wedge \mathrm{d}\,x^j.$$

The previous expression contain the basis $\omega^\sigma_I \wedge \mathrm{d}\,x^i$, $|I| = r$ only in the case when $I = Jj$, $|J| = r-1$. The condition $\mathrm{p}_1\,\mathrm{d}\,\alpha = 0$ then leads to $A^J_\sigma = 0$ for $|J| = r-1$. Continuing in the same manner we arrive at $A^J_\sigma = 0$, $0 \le |J| \le r-1$, and consequently $\alpha = 0$. The condition $\mathrm{d}\,\alpha = 0$ implies $\mathrm{p}_1\,\mathrm{d}\,\alpha = 0$ which in turn implies $\alpha = 0$ as required for exactness.

Now, we prove exactness of (V.20) at Θ^r_q, $2 \le q \le n$. We express an element $\alpha \in \Omega^r_q W$ in some fibered chart as

$$\alpha = \omega^\sigma_J \wedge A^J_\sigma + \mathrm{d}(\omega^\nu_I \wedge B^I_\nu), \tag{V.24}$$

where $0 \le |I|, |J| \le r-1$. Again we set

$$\mathrm{p}_1\,\mathrm{d}\,\alpha = \mathrm{d}\,\omega^\sigma_J \wedge \mathrm{h}\,A^J_\sigma - \omega^\sigma_J \wedge \mathrm{h}\,\mathrm{d}\,A^J_\sigma = 0.$$

We decompose the differential forms A^J_σ into two parts as follows

$$A^J_\sigma = a^J_\sigma + b^J_\sigma,$$

where the forms a^J_σ are generated by ω^ν_I, $0 \le |I| \le r-1$ while the forms b^J_σ do not contain any factors of this type, i.e. of the type

$$b^J_\sigma = c^J_{\sigma\ i_1 \dots i_q}\,\mathrm{d}\,x^{i_1} \wedge \dots \wedge \mathrm{d}\,x^{i_q} + c^{JI_1}_{\sigma\sigma_1\ i_2 \dots i_q}\,\mathrm{d}\,y^{\sigma_1}_{I_1} \wedge \mathrm{d}\,x^{i_2} \wedge \dots \wedge \mathrm{d}\,x^{i_q} +$$
$$+ c^{JI_1I_2}_{\sigma\sigma_1\sigma_2\ i_3 \dots i_q}\,\mathrm{d}\,y^{\sigma_1}_{I_1} \wedge \mathrm{d}\,y^{\sigma_2}_{I_2} \wedge \mathrm{d}\,x^{i_3} \wedge \dots \wedge \mathrm{d}\,x^{i_q} + \dots + c^{JI_1 \dots I_q}_{\sigma\sigma_1 \dots \sigma_q}\,\mathrm{d}\,y^{\sigma_1}_{I_1} \wedge \dots \wedge \mathrm{d}\,y^{\sigma_q}_{I_q},$$

where $|I_1| = \dots = |I_q| = r$. Setting $\mathrm{p}_1\,\mathrm{d}\,\alpha = 0$, we obtain

$$\omega^\sigma_{Jj} \wedge \mathrm{d}\,x^j \wedge \mathrm{h}\,b^J_\sigma + \omega^\sigma_J \wedge \mathrm{h}\,\mathrm{d}\,b^J_\sigma = 0.$$

Again, we consider terms containing the basis ω_I^σ, with $|I| = r$. Such terms must vanish separately and we get

$$\mathrm{d}\, x^{(j} \wedge \mathrm{h}\, b_\sigma^{J)} = \mathrm{h}(\mathrm{d}\, x^{(j} \wedge b_\sigma^{J)}) = 0,$$

for $|J| = r-1$. By solving this system of equations we obtain that the form b_σ^J is generated by the 2-forms $\mathrm{d}\,\omega_I^\sigma$, $|I| = r-1$. Thus, $\mathrm{h}\, A_\sigma^J = \mathrm{h}(a_\sigma^J + b_\sigma^J) = 0$ for $|J| = r-1$. This implies that $\mathrm{h}\,\mathrm{d}\, A_\sigma^J = 0$ for $|J| = r-1$ and the resulting expression coincides with (V.24) expect that the summation is taken over J such that $|J| \leq r-2$. By repeating this procedure, we get that

$$\mathrm{h}\, A_\sigma^J = 0$$

for all $|J| \leq r-1$. Suppose that $\mathrm{d}\,\alpha = 0$. Then also $\mathrm{h}\,\mathrm{d}\,\alpha = 0$ and $\alpha = \alpha' + \mathrm{d}\,\beta$, where $\beta \in \Theta_{q-1}^r$ and the order of contactness of α' is greater than one by virtue of Theorem III.4. The condition $\mathrm{d}\,\alpha = 0$ implies that $\mathrm{d}\,\alpha' = 0$ and via Theorem V.4 it must be that $\alpha = \mathrm{d}(A\alpha' + \beta)$. But $\mathrm{h}\, A\alpha' = A\, \mathrm{p}_1\, \alpha' = 0$ due to the higher contactness of α'. Consequently, $A\alpha' + \beta \in \Theta_{q-1}^r$.

Finally, we prove exactness of the sequence (V.20) of Θ_q^r, where $n+1 \leq q \leq P$. Let $\alpha \in \Theta_q^r$, $\alpha = \alpha' + \mathrm{d}\,\beta$, where $\alpha' \in \Omega_{q,\mathrm{c}}^r$ and $\beta \in \Omega_{q-1,\mathrm{c}}^r$. Again we suppose that $\mathrm{d}\,\alpha = 0$. Then also $\mathrm{d}\,\alpha' = 0$ and $\alpha' = \mathrm{d}\, A\alpha'$. Since $\mathrm{p}_{q-n}\, \alpha' = 0$, we have $\mathrm{p}_{q-1-n}(A\alpha' + \beta) = A\, \mathrm{p}_{q-n}\, \alpha' = 0$. This means that $A\alpha' + \beta \in \Omega_{q-1,\mathrm{c}}^r$ and proves the exactness at Θ_q^r. □

Lemma V.7. *For every q, $1 \leq q \leq P$, the sheaf Θ_q^r is soft.*

Proof. All sheaves $\Omega_{q,\mathrm{c}}^r$ can be considered as sheaves of Ω_0^r-modules. Thus every partition of unity on the manifold Y subordinate to an open covering (U_ι) induces a sheaf partition of unity subordinate to the same covering. Thus, the sheaf $\Omega_{q,\mathrm{c}}^r$ is fine hence soft.

Consider the sheaf morphism $\mathrm{d}\colon \Omega_{1,\mathrm{c}}^r \to \Omega_2^r$ and the corresponding short exact sequence

$$0 \longrightarrow \ker \mathrm{d} \longrightarrow \Omega_{1,\mathrm{c}}^r \longrightarrow \operatorname{im} \mathrm{d} = \mathrm{d}\,\Omega_{1,\mathrm{c}}^r \longrightarrow 0.$$

Since the sheaf $\Omega_{1,\mathrm{c}}^r$ is soft and $\ker \mathrm{d} = 0$, the quotient sheaf $\mathrm{d}\,\Omega_{1,\mathrm{c}}^r$ must also be soft. We continue by induction on q and suppose that $\mathrm{d}\,\Omega_{q-1,\mathrm{c}}^r$ is soft. For $q \geq 2$ we consider the sheaf morphism $\mathrm{d}\colon \Omega_{q,\mathrm{c}}^r \to \Omega_{q+1}^r$ and the corresponding short exact sequence

$$0 \longrightarrow \ker \mathrm{d} \longrightarrow \Omega_{q,\mathrm{c}}^r \longrightarrow \operatorname{im} \mathrm{d} = \mathrm{d}\,\Omega_{q,\mathrm{c}}^r \longrightarrow 0.$$

By Lemma (V.6) it holds $\ker \mathrm{d} = \operatorname{im} \mathrm{d}$ and so, since $\mathrm{d}\,\Omega_{q-1,\mathrm{c}}^r$ is soft by hypothesis, the quotient sheaf $\mathrm{d}\,\Omega_{q,\mathrm{c}}^r$ must also be soft. The sum of soft subsheaves $\Theta^r_q = \mathrm{d}\,\Omega^r_{q-1,\mathrm{c}} + \Omega^r_{q,\mathrm{c}}$ is soft as can be seen from the following short exact sequence

$$0 \longrightarrow \ker \mathrm{d} \xrightarrow{\ \chi\ } \Omega^r_{q-1,\mathrm{c}} \oplus \Omega^r_{q,\mathrm{c}} \xrightarrow{\ \psi\ } \mathrm{d}\,\Omega^r_{q-1,\mathrm{c}} + \Omega^r_{q,\mathrm{c}} \longrightarrow 0,$$

where $\oplus$ denotes the direct sum of sheaves of modules and

$$\chi\colon \eta \to (\eta, -\mathrm{d}\,\eta)$$
$$\psi\colon (\eta, \rho) \to \mathrm{d}\,\eta + \rho.$$

$\square$

2.3. The variational sequence. We shall construct the quotient sequence of the de Rham sequence of differential forms by the subsequence (V.20) as depicted in Figure 1, where the vertical arrows are understood as natural inclusions and natural quotient mappings. The sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^r_0 \xrightarrow{\ \mathscr{E}_0\ } \Omega^r_1/\Theta^r_1 \xrightarrow{\ \mathscr{E}_1\ } \Omega^r_2/\Theta^r_2 \xrightarrow{\ \mathscr{E}_2\ } \cdots$$

$$\cdots \xrightarrow{\ \mathscr{E}_{P-1}\ } \Omega^r_P/\Theta^r_P \xrightarrow{\ \mathscr{E}_P\ } \Omega^r_{P+1} \xrightarrow{\ \mathscr{E}_{P+1}\ } \cdots \xrightarrow{\ \mathscr{E}_{N-1}\ } \Omega^r_N \xrightarrow{\ \mathscr{E}_N\ } 0,$$

defined by Figure 1, is called the *variational sequence of order* r over Y. Clearly, $\mathscr{E}_{P+1} = \mathrm{d}, \dots, \mathscr{E}_N = \mathrm{d}$.

Theorem V.8. *The variational sequence of order r is an acyclic resolution of the constant sheaf $\mathbb{R}$ over Y.*

Proof. We proved in Lemma V.6 that the first row is an exact subsequence of the second row. Thus the third row, i.e. the quotient sequence, is also exact, and is a resolution of $\mathbb{R}$. It suffices to show that this resolution is soft. The first row is soft by Lemma V.7, the second row is fine hence soft by construction. The short exact sequences in the columns guarantee the softness of the quotient sequence. $\square$

The variational sequence is also denoted by

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathscr{V}^r.$$

Let $\Gamma(Y, \mathscr{V}^r)$ be the complex of global sections

$$0 \longrightarrow \Gamma(Y, \mathbb{R}) \longrightarrow \Gamma(Y, \Omega^r_0) \longrightarrow \Gamma(Y, \Omega^r_1/\Theta^r_1) \to \Gamma(Y, \Omega^r_2/\Theta^r_2) \to \cdots$$

$$\cdots \to \Gamma(Y, \Omega^r_P/\Theta^r_P) \to \Gamma(Y, \Omega^r_{P+1}) \longrightarrow \cdots \longrightarrow \Gamma(Y, \Omega^r_N) \longrightarrow 0.$$

We are lead to the following result concerning the cohomology groups of this complex.

$$
\begin{tikzcd}
 & & & \{0\} \arrow[d] & \{0\} \arrow[d] & & \{0\} \arrow[d] & & \{0\} \arrow[d] & & & & \\
 & & \{0\} \arrow[r] \arrow[d] & \Theta_1^r \arrow[r, "\mathrm{d}_1"] \arrow[d] & \Theta_2^r \arrow[r, "\mathrm{d}_2"] \arrow[d] & \cdots \arrow[r, "\mathrm{d}_{q-1}"] & \Theta_q^r \arrow[r, "\mathrm{d}_q"] \arrow[d] & \cdots \arrow[r, "\mathrm{d}_{P-1}"] & \Theta_P^r \arrow[r, "\mathrm{d}_P"] \arrow[d] & \{0\} \arrow[d] & & & \\
\{0\} \arrow[r] & \mathbb{R} \arrow[r] & \Omega_0^r \arrow[r, "\mathrm{d}_0"] \arrow[dr, "E_0"'] & \Omega_1^r \arrow[r, "\mathrm{d}_1"] \arrow[d] & \Omega_2^r \arrow[r, "\mathrm{d}_2"] \arrow[d] & \cdots \arrow[r, "\mathrm{d}_{q-1}"] & \Omega_q^r \arrow[r, "\mathrm{d}_q"] \arrow[d] & \cdots \arrow[r, "\mathrm{d}_{P-1}"] & \Omega_P^r \arrow[r, "\mathrm{d}_P"] \arrow[d] & \Omega_{P+1}^r \arrow[r, "\mathrm{d}_{P+1}", "E_{P+1}"'] & \cdots \arrow[r, "\mathrm{d}_{N-1}", "E_{N-1}"'] & \Omega_N^r \arrow[r, "\mathrm{d}_N", "E_N"'] & \{0\} \\
 & & & \Omega_1^r/\Theta_1^r \arrow[r, "E_1"'] \arrow[d] & \Omega_2^r/\Theta_2^r \arrow[r, "E_2"'] \arrow[d] & \cdots \arrow[r, "E_{q-1}"'] & \Omega_q^r/\Theta_q^r \arrow[r, "E_q"'] \arrow[d] & \cdots \arrow[r, "E_{P-1}"'] & \Omega_P^r/\Theta_P^r \arrow[d] \arrow[ur, "E_P"'] & & & & \\
 & & & \{0\} & \{0\} & & \{0\} & & \{0\} & & & &
\end{tikzcd}
$$

Figure 1. The variational sequence

Theorem V.9. *For every $q \geq 0$*

$$H^q(\Gamma(Y, \mathscr{V}^r)) = H^q(Y, \mathbb{R}).$$

Proof. This follows immediately from Theorem V.8 and the abstract de Rham Theorem V.3. □

2.4. Representation of the Variational Sequence. When dealing with the variational sequence it is sometimes of advantage to pick some special representatives in the quotient sheaves Ω^r_q/Θ^r_q, which are themselves elements of some possibly higher order sheaves Ω^s_q. The following Lemma guarantees the success of our efforts.

Lemma V.10. *Let $0 < r < s$ be integers. Then the scheme*

$$\begin{tikzcd}
0 \arrow[r] & \Theta^r_q \arrow[r, hook] \arrow[d, "(\pi^{s,r})^*"'] & \Omega^r_q \arrow[r] \arrow[d, "(\pi^{s,r})^*"'] & \Omega^r_q/\Theta^r_q \arrow[r] \arrow[d, "Q^{s,r}_q"'] & 0 \\
0 \arrow[r] & \Theta^s_q \arrow[r, hook] & \Omega^s_q \arrow[r] & \Omega^s_q/\Theta^s_q \arrow[r] & 0
\end{tikzcd}$$

commutes and the mapping $Q^{s,r}_q$ is injective and is defined uniquely.

Proof. The statement is the consequence of Lemma V.1. □

Two elements ρ, η of the sheaf of differential forms Ω^r_q belonging to the same class Ω^r_q/Θ^r_q are called *equivalent*. Two elements $\rho \in \Omega^r_q$ and $\eta \in \Omega^s_q$ are called *equivalent in the generalized sense* if there exist an integer t, such that the images $(\pi^{t,r})^*\rho$ and $(\pi^{t,s})^*\eta$ belong to the same class Ω^t_q/Θ^t_q. In such a case we write $\rho \sim \eta$.

The situation is simple for $1 \leq q \leq n = \dim X$. Here, we obtain by definition $\Omega^r_q/\Theta^r_q \cong \operatorname{im} \mathrm{h}$. The classes in Ω^r_q/Θ^r_q are represented by elements of sheaves of differential forms $\mathrm{h}\,\rho \in \Omega^{r+1}_q$.

In the case of elements of the sheaves Ω^r_q/Θ^r_q, where $n+1 \leq q \leq P$, we apply the results from Chapter IV. To this end, let $k = q - n - 1$.

Theorem V.11. *The interior Euler mapping $I : \Omega^r_{n+k}W \to \Omega^{2r+1}_{n+k}W$ has the following properties*

(a) *The relation $I(\rho) \sim \rho$ is fulfilled in the generalized sense, meaning that*

$$\left(\pi^{2r+1,r}\right)^* \rho - I(\rho) \in \Theta^{2r+1}_k. \tag{V.25}$$

(b) *The operator I is a projection in the following sense*

$$I^2 = I \circ \left(\pi^{2r+1,r}\right)^*. \tag{V.26}$$

(c) The operator I fulfills the relation

$$I \circ \mathrm{d} \circ I \circ \mathrm{d}\, \rho = 0. \tag{V.27}$$

Proof. (a) follows from Theorem IV.2 (see also Remark 7).
(b) We apply the operator I to equation (V.25). Using Lemma IV.3, we prove that $I(\eta) = 0$ for every $\eta \in \Theta^{2r+1}_{n+k}$: η is of the form $\eta = \alpha + \mathrm{d}\,\beta$ where $\alpha \in \Omega^{2r+1}_{n+k,\mathrm{c}}$ and $\beta \in \Omega^{2r+1}_{n+k-1,\mathrm{c}}$. It is evident that $I(\alpha) = 0$ Moreover, $I(\mathrm{d}\,\beta) = 0$ by applying Lemma IV.3.
(c)Let $\eta = \mathrm{p}_{k+1}\,\mathrm{d}\,\rho$. Then from Theorem IV.2 we have

$$\mathrm{p}_{k+1}\,\mathrm{d}\,\mathrm{p}_k\,\rho + \mathrm{p}_{k+1}\,\mathrm{d}\,\mathrm{p}_{k+1}\,\rho = \mathrm{p}_{k+1}\,\mathrm{d}\,\rho = I(\mathrm{p}_{k+1}\,d\rho) + \mathrm{p}_{k+1}\,\mathrm{d}\,\mathrm{p}_{k+1}\,\zeta|_W\,.$$

To this equation we apply first the operator $\mathrm{p}_{k+2}\,\mathrm{d}$ and obtain

$$0 = \mathrm{p}_{k+2}\,\mathrm{d}\,\mathrm{p}_{k+1}\,\mathrm{d}\,\mathrm{p}_k\,\rho = \mathrm{p}_{k+2}\,\mathrm{d}\,I\,\mathrm{p}_{k+1}\,\mathrm{d}\,\rho + \mathrm{p}_{k+2}\,\mathrm{d}\,\mathrm{p}_{k+1}\,\mathrm{d}\,\mathrm{p}_{k+1}\,(\zeta|_W - \rho)\,.$$

The left hand side of this equation vanishes identically and on the right hand side we obtain

$$\mathrm{p}_{k+2}\,\mathrm{d}\,\mathrm{p}_{k+1}\,\mathrm{d}\,\mathrm{p}_{k+1}\,(\zeta|_W - \rho) = -\,\mathrm{p}_{k+2}\,\mathrm{d}\,\mathrm{p}_{k+1}\,\mathrm{d}\,\mathrm{p}_{k+1}\,(\zeta|_W - \rho)$$

and get

$$\mathrm{p}_{k+2}\,\mathrm{d}\,I\,\mathrm{p}_{k+1}\,\mathrm{d}\,\rho = \mathrm{p}_{k+2}\,\mathrm{d}\,\mathrm{p}_{k+2}\,\mathrm{d}\,\mathrm{p}_{k+1}\,\zeta.$$

Now we once more apply I, use the Lemma IV.3 and obtain

$$I\,\mathrm{p}_{k+2}\,\mathrm{d}\,I\,\mathrm{p}_{k+1}\,\mathrm{d}\,\rho = 0.$$

□

We denote the sheaf morphisms corresponding to h and I by the same letters and call the morphisms

$$\Omega^r_q/\Theta^r_q \ni [\rho] \to \begin{cases} \mathrm{h}\,\rho \in \Omega^{r+1}_q & \text{for } 1 \le q \le n, \\ I\rho \in \Omega^{2r+1}_q & \text{for } n+1 \le q \le P \end{cases} \tag{V.28}$$

the *representation of the variational sequence* by sheaves of differential forms. Consider the diagram in the Figure 2 which depicts variational sequences of lower order, denoted by full arrows, included into variational sequences of higher order, denoted by dashed and dotted arrows. The quotient mappings are denoted by double arrows, the representation morphisms are denoted by triple arrows. Inclusions are denoted by hooked arrows. Similar diagrams could be drawn for the horizontalization mapping h. The diagram illustrates the behavior of the projection operator I with respect to the canonical inclusion of lower order variational sequences into higher order variational sequences.

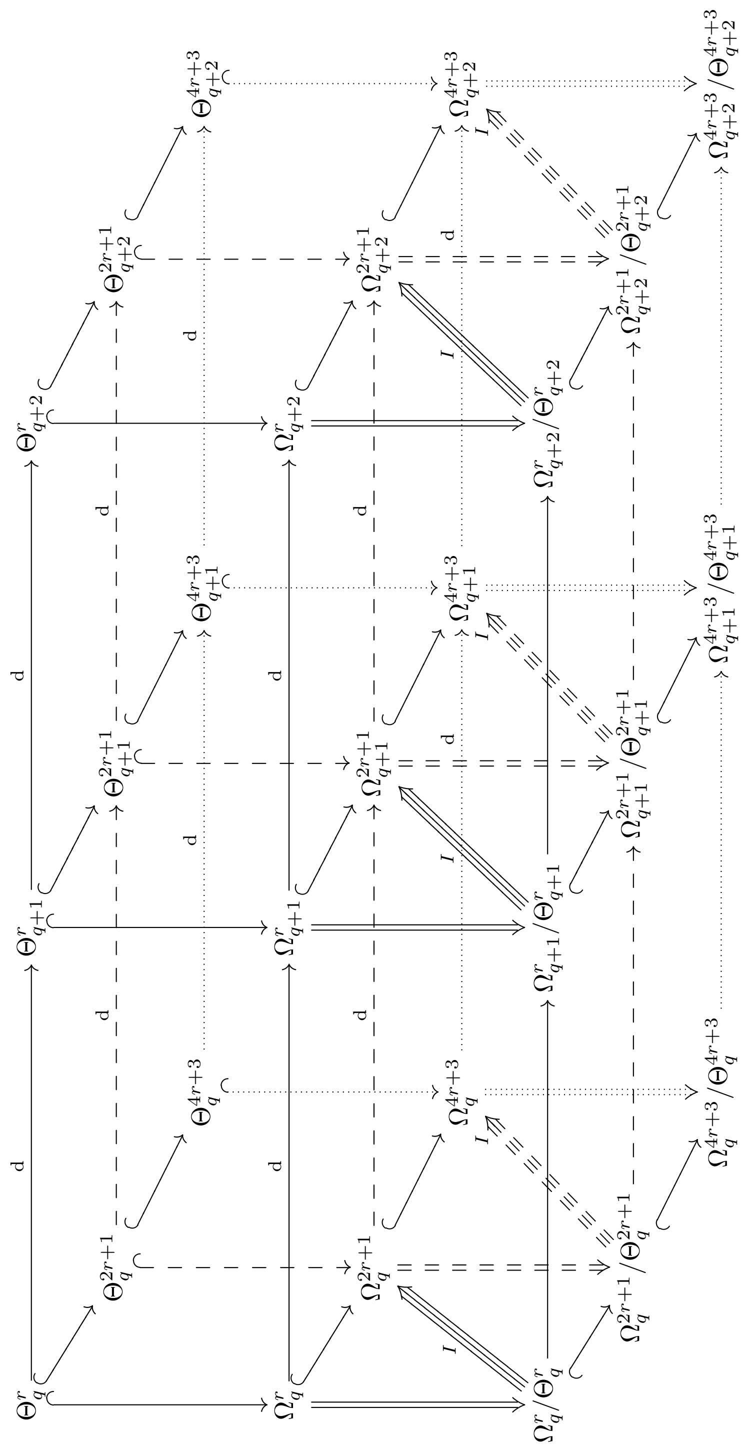

Figure 2. The representation of the variational sequence ($q > n$)

Remark 8. There arises the natural question, how to recognize whether a differential form $\eta \in \Omega^s_q W$ is a representative. This question is easily answered, because, in some sense, the representation mappings behave like projectors (see Remark 3 and Theorem V.11). The form η must therefore fulfill the condition $\mathrm{h}\,\eta = (\pi^{s+1,s})^*\eta$ for $1 \leq q \leq n$ resp. $I\eta = (\pi^{2s+1,s})^*\eta$ for $n < q \leq P$.

2.5. Examples. We now consider some examples of representatives that have some meaning in the calculus of variations. to this end, let $\rho \in \Omega^r_q W$ be a differential q-form expressed in some fibered chart.

– Let $\rho \in \Omega^r_n W$ be a differential n-form. Then $[\rho]$, its class, is represented by the horizontal form $\mathrm{h}\,\rho \in \Omega^{r+1}_n W$ which is polynomial in the coordinates y^σ_J, $|J| = r+1$, as seen from (III.14). These representatives are customarily called Lagrangians and in local coordinates we have

$$L_\rho = \mathrm{h}\,\rho = \varepsilon^{i_1\dots i_n} \sum_{s=0}^{n} P^{J_1\dots J_s}_{\sigma_1\dots\sigma_s\; i_{s+1}\dots i_n} y^{\sigma_1}_{J_1 i_1} \cdots y^{\sigma_s}_{J_s i_s} \omega_0. \tag{V.29}$$

– Let $\rho \in \Omega^r_{n+1} W$ be a differential $(n+1)$-form. Then $[\rho]$, its class, is represented by the horizontal form $I\rho \in \Omega^{2r+1}_{n+1} W$ which is polynomial in the coordinates y^σ_J, $|J| \geq r+1$, as can be seen from (III.14) and the formula (II.12) for formal derivatives. These representatives are called dynamical forms and their local expressions are

$$E_\rho = I\rho = \sum_{|J_1|=0}^{r} (-1)^{|J_1|} \varepsilon^{i_2\dots i_{n+1}}\, \mathrm{D}_{J_1} \left(\sum_{s=1}^{q} s P^{J_1\dots J_s}_{\sigma_1\dots\sigma_s\; i_{s+1}\dots i_q} y^{\sigma_2}_{J_2 i_2} \cdots y^{\sigma_s}_{J_s i_s} \right) \omega^{\sigma_1} \wedge \omega_0. \tag{V.30}$$

– Let $\rho \in \Omega^r_{n+2} W$ be a differential $(n+2)$-form. Then $[\rho]$, its class, is represented by the horizontal form $I\rho \in \Omega^{2r+1}_{n+2} W$ which is polynomial in the coordinates y^σ_J, $|J| \geq r+1$, as can be seen from (III.14) and the formula (II.12) for formal derivatives. These representatives are called Helmholtz-Sonin forms and their local expressions are

(V.31) $$H_\rho = I\rho = \tfrac{1}{2}\sum_{a=0}^{2r}\left[\sum_{b=0}^{a}\sum_{c=a-b}^{r}(-1)^c\tbinom{c}{a-b}\varepsilon^{i_3\dots i_{n+2}}\,\mathrm{D}_{j_{a+1}\dots j_{b+c}}\right.$$
$$\left.\left(\sum_{s=2}^{q}\tbinom{s}{2}P^{(j_1\dots j_b)}_{\sigma_1}{}^{(j_{b+1}\dots j_{b+c})}_{\sigma_2}{}^{J_3\dots J_s}_{\sigma_3\dots\sigma_s\,i_{s+1}\dots i_q}\,y^{\sigma_3}_{J_3 i_3}\cdots y^{\sigma_s}_{J_s i_s}\right)\right]\omega^{\sigma_1}_{(j_1\dots j_a)}\wedge\omega^{\sigma_2}\wedge\omega_0.$$

We conclude with examples of mappings induced by the variational sequence. these are precisely the classical mappings known in the calculus of variations.

– Let $\rho \in \Omega^r_n W$ be an n-form. We compute the representative of the class $[\mathrm{d}\,\rho]$ given by $I\,\mathrm{d}\,\mathrm{h}\,\rho$. In local coordinates we obtain by using (V.29) and (V.30)

(V.32)
$$I\,\mathrm{d}\,\mathrm{h}\,\rho = \sum_{|J|=0}^{r}(-1)^{|J|}\,\mathrm{D}_J\frac{\partial}{\partial y^\nu_J}\left[\varepsilon^{i_1\dots i_n}\sum_{s=0}^{n}P^{J_1\dots J_s}_{\sigma_1\dots\sigma_s\,i_{s+1}\dots i_n}y^{\sigma_1}_{J_1 i_1}\cdots y^{\sigma_s}_{J_s i_s}\right]\omega^\nu\wedge\omega_0.$$

We see that for a horizontal form $\lambda = \mathfrak{L}\omega_0$ we recover the Euler-Lagrange expressions associated with λ

(V.33) $$I\,\mathrm{d}\,\lambda = \sum_{|J|=0}^{r}(-1)^{|J|}\,\mathrm{D}_J\frac{\partial\mathfrak{L}}{\partial y^\nu_J}\omega^\nu\wedge\omega_0.$$

– Let $\rho \in \Omega^r_{n+1}W$ be an $(n+1)$-form. The representative of the class $[\mathrm{d}\,\rho]$ is given by $I\,\mathrm{d}\,\rho$.

(V.34) $$I\,\mathrm{d}\,I\rho = \tfrac{1}{2}\sum_{a=0}^{2r}\left[\left(\frac{\partial\mathfrak{E}_\nu}{\partial y^\sigma_{j_1\dots j_a}} - (-1)^a\frac{\partial\mathfrak{E}_\sigma}{\partial y^\nu_{j_1\dots j_a}} - \right.\right.$$
$$\left.\left. - \sum_{b=a+1}^{r}(-1)^b\tbinom{b}{a}\,\mathrm{D}_{j_{a+1}\dots j_b}\frac{\partial\mathfrak{E}_\sigma}{\partial y^\nu_{j_1\dots j_b}}\right)\omega^\sigma_{j_1\dots j_a}\right]\wedge\omega^\nu\wedge\omega_0,$$

where the coefficients $\mathfrak{E}^\sigma$ are the components of E_ρ in the formula (V.30)

(V.35) $$E_\rho = \mathfrak{E}_\sigma\omega^\sigma\wedge\omega_0.$$

We can also consider an arbitrary dynamical form (V.35) as a special case and obtain formally identical representatives to (V.34).

Bibliography

[1] S. J. Aldersley: Higher Euler operators and some of their applications, J. Math. Phys. 20 (1979), 522–531.

[2] I. M. Anderson, T. Duchamp: On the existence of global variational principles, Am. J. Math. 102 (1980), 781–867.

[3] I. M. Anderson: Introduction to the variational bicomplex, Contemporary Mathematics 132 (1992), 51–73.

[4] I. M. Anderson: *The Variational Bicomplex*, preprint, Utah State University, 1989.

[5] I. M. Anderson, J. Pohjanpelto: The cohomology of invariant variational bicomplexes, in: *Geometric and Algebraic Methods in Differential Equations*, Proc. Conf., University of Twente, June 1993.

[6] P. Dedecker: On the generalization of symplectic geometry to multiple integrals in the calculus of variations, in: Lecture Notes in Math. 570, Springer, Berlin, 1977, 395–456.

[7] P. Dedecker, W. M. Tulczyjew: Spectral sequences and the inverse problem of calculus of variations, in: *Differential Geometric Methods in Mathematical Physics*, Proc. Internat. Coll., Aix-en-Provence, France, 1979; Lecture Notes in Math. 836, Springer, Berlin, 1980, 498–503.

[8] M. Francaviglia, M. Palese, R. Vitolo: Superpotentials in variational sequences, in: *Differential Geometry and its Applications*, Proc. Conf., Brno, Czech Republic, August 1998, I. Kolář, D. Krupka, J. Slovák, eds.; Masaryk University, Brno, 1999, 469–480.

[9] M. Francaviglia, M. Palese, R. Vitolo: Second order variations in variational sequences, in: *Colloq. on Diff. Geom.*, Proc. Conf., Debrecen, Hungary, July 2000, L. Kozma, ed.; Debrecen, 2001.

[10] P. L. García: The Poincaré-Cartan invariant in the calculus of variations, Symposia Mathematica 14 (1974), 219–246.

[11] M. Gotay: An exterior differential systems approach to the Cartan form, in: Géometrie Symplectique et Physique Mathématique, P. Donato et all., eds., Birkhäuser, Boston, 1991.

[12] D. R. Grigore, D. Krupka: Invariants of velocities and higher-order Grassmann bundles, Journ. Geom. Phys. 24 (1998), 244–264.

[13] D. R. Grigore: The variational sequence on finite jet bundle extensions and the Lagrange formalism, Diff. Geom. Appl. 10 (1999), 43–77.

[14] D. R. Grigore: Variationally trivial lagrangians and locally variational differential equations of arbitrary order, Diff. Geom. Appl. 10 (1999), 79–15.

[15] Jana Kašparová: A representation of the first order variational sequence in field theory, in: *Differential Geometry and its Applications*, Proc. Conf., Brno, Czech Republic, August 1998, I. Kolář, O. Kowalski, D. Krupka, J. Slovák, eds.; Masaryk University, Brno, 1999, 493–502.

[16] Jana Kašparová: *Variational sequences in field theory*, Ph.D. thesis, Mathematical Institute, Silesian University, Opava, 2000.

[17] Jana Kašparová: On some global representants of the 1-st order variational sequence, Preprint series in Global Analysis GA 4/2000, Mathematical Institute, Silesian University, Opava, 2000.

[18] L. Klapka: Euler–Lagrange expressions and closed two-forms in higher order mechanics, in: *Geometrical Methods in Physics*, Proc. Conf., Vol 2, Nové Město na Moravě, Czechoslovakia, 1983, D. Krupka, ed.; J. E. Purkyně University, Brno, 1984, 149–453.

[19] I. Kolář, P. Michor, J. Slovák: *Natural Operations in Differential Geometry*, Springer Verlag, Berlin, Heidelberg, 1993.

[20] M. Krbek: *Variational sequence in mechanics and field theory*, Diploma Thesis, Faculty of Science, Masaryk University, Brno, 1998.

[21] M. Krbek, J. Musilová, J. Kašparová: The representation of the variational sequence in field theory, in: *Proc. Colloq. on Diff. Geom.*, Debrecen, Hungary, July 2000, L. Kozma, ed.; Debrecen, 2001.

[22] M. Krbek, J. Musilová: Representation of the variational sequence, submitted to Reports on Mathematical Physics, Torun, 2002.

[23] D. Krupka: Some Geometric Aspects of the Calculus of Variations in Fibered Manifolds, Folia Fac. Sci. Nat. UJEP Brunensis 14 (1973).

[24] D. Krupka: On the structure of EulerLagrange mapping, Arch. Math. (Brno) 10 (1974), 55–61.

[25] D. Krupka: On generalized invariant transformations, Rep. Math. Phys. (Torun) 5 (1974), 353–358.

[26] D. Krupka: A geometric theory of ordinary first order variational problems in fibered manifolds, I. Critical sections, J. Math. Anal. Appl. 49 (1975), 180-206.

[27] D. Krupka: A map associated to the Lepagean forms of the calculus of variations in fibered manifolds, Czechoslovak Math. J. 27 (1977), 114–118.

[28] D. Krupka: On the local structure of EulerLagrange mapping of the calculus of variations, in: *Differential Geometry and its Applications*, Proc. Conf., Nové Město na Moravě, Czechoslovakia, 1980, O. Kowalski, ed.; Universita Karlova, Praha, 1981, 181–188.

[29] D. Krupka: Natural Lagrangian structures, Semester on Differential Geometry, Banach Center, Warsaw, 1979; Banach Center Publications 12 (1984), 185–210.

[30] D. Krupka, O. Stěpánková: On the Hamilton form in second order calculus of variations, Proc. of the Meeting *Geometry and Physics*, Florence, October 1982; Pitagora Editrice Bologna, 1983, 85–101.

[31] D. Krupka, J. Musilová: Hamilton extremals in higher order mechanics, Arch. Math. (Brno) 20 (1984), 21–30.

[32] D. Krupka: Regular lagrangians and Lepagean forms, in: *Differential Geometry and its Applications*, Proc. Conf., Brno, Czechoslovakia, 1986, D. Krupka, A. Švec, eds.; D. Reidel, Dordrecht, 1986, 111–148.

[33] D. Krupka: Geometry of Lagrangean structures 2, 3, Arch. Math. (Brno) 22 (1986), 211-228; *14th Winter school on Abstract Analysis*, Proc. Conf., Srní, Czechoslovakia, January 1986; Suppl. ai rend. del Circ. Mat. di Palermo 14 (1987), 178–224.

[34] D. Krupka: Variational sequences on finite order jet spaces, in: *Differential Geometry and its Applications*, Proc. Conf., Brno, Czechoslovakia, 1989, J. Janyška, D. Krupka, eds.; World Scientific, Singapore, 1990, 236–254.

[35] D. Krupka, J. Janyška: *Lectures on Differential Invariants*, J. E. Purkyně University, Brno, Czechoslovakia, 1990.

[36] D. Krupka: Topics in the calculus of variations: Finite order variational sequences, in: *Differential Geometry and its Applications*, Proc. Conf., Opava, Czechoslovakia, 1992; Silesian University, Opava, 1993, 473–495.

[37] D. Krupka: Lectures on variational sequences, Open Education & Sciences, Opava, Czech Republic, 1995.

[38] D. Krupka: The contact ideal, Diff. Geom. Appl. 5 (1995), 257-276.

[39] D. Krupka: The trace decomposition problem, Beitr. Algebra Geom. 36 (1995), 303–315.

[40] D. Krupka: *The Geometry of Lagrange structures*, Preprint Series in Global Analysis GA 7/1997, Mathematical Institute, Silesian University, Opava 1997.

[41] D. Krupka: Variational sequence in mechanics, Calc. Var. 5 (1997), 558–583.

[42] D. Krupka: Geometry of Lagrange Structures, II. Elementary Sheaf Theory, Preprint Series in Global Analysis GA 2/1998, Mathematical Institute, Silesian University, Opava, 1998.

[43] D. Krupka: J. Musilová, Trivial lagrangians in field theory, Diff. Geom. Appl. 9 (1998), 293–305.

[44] D. Krupka: J. Musilová, Recent results in variational sequence theory, in: *Proc. Colloq. on Diff. Geom.*, Debrecen, Hungary, July 2000, L. Kozma, ed.; Debrecen 2001.

[45] D. Krupka: Variational sequences and variational bicomplexes, in: *Differential Geometry and its Applications*, Proc. Conf., Brno, Czechoslovakia, August 1998, I. Kolář, O. Kowalski, D. Krupka, J. Slovák, eds.; Masaryk University, Brno, 1999, 525–531.

[46] O. Krupková: Higher order constraint system on fibered manifolds: An exterior differential systems approach, in: *New Developments in Differential Geometry*, Proc. Conf., Debrecen, Hungary, 1994, J. Tamassy, J. Szenthe, eds.; Kluwer, Dordrecht, 1996, 255-278.

[47] O. Krupková: *The Geometry of Ordinary Variational Equations*, Lecture Notes in Math. 1678, Springer, Berlin, 1997.

[48] O. Krupková: Hamiltonian field theory revisited: A geometric approach to regularity, in: *Proc. Colloq. on Diff. Geom.*, Debrecen, Hungary, July 2000, L. Kozma, ed.; Debrecen 2001.

[49] J. Muñoz Masqué: Poincaré-Cartan forms in higher order variational calculus on fibered manifolds, Revista Matemática Iberoamericana, Vol. 1, 4, 1985, 85–126.

[50] J. Musilová: Variational sequence in higher order mechanics, in: *Differential Geometry and its Applications*, Proc. Conf., Brno, Czech Republic, September 1995, J. Janyška, I. Kolář, J. Slovăk, eds.; Masaryk University, Brno, 1996, 611–624.

[51] J. Musilová, M. Krbek: A note to the representation of the variational sequence in mechanics, in: *Differential Geometry and its Applications*, Proc. Conf., Brno, Czech Republic, August 1998, I. Kolář, O. Kowalski, D. Krupka, J. Slovák, eds.; Masaryk University, Brno, 1999, 511–523.

[52] J. Novotný: On the geometric foundations of the Lagrange formulation of general relativity, in: *Differential Geometry*, Coll. Math. Soc. J. Bolyai 31, Budapest, Hungary, 1979; North-Holland, Amsterdam, 1982.

[53] P. J. Olver: The canonical contact form, preprint, University of Minnesota, 2000.

[54] J. F. Pommaret: *Partial Differential Equations and Group Theory*, Kluwer Academic Publishers, Dordrecht, 1994.

[55] J. F. Pommaret: Spencer sequence and variational sequence, Acta Appl. Math. 41 (1995), 285–296.

[56] H. Rund: Integral formulae associated with EulerLagrange operator of mul- tiple integral problems in the calculus of variations, Aequationes Math. 11 (1974), 212–229.

[57] D. Saunders: *The Geometry of Jet Bundles*, Cambridge Univ. Press, 1989.

[58] G. Sardanashvily: *Generalized Hamiltonian Formalism for Field Theory*, World Scientific, Singapore, 1995.

[59] G. Sardanashvily: Differential geometry of composite manifolds, preprint

[60] W. F. Shadwick: The Hamiltonian formulation of regular r-th order Lagrangian field theories, Letters in Math. Phys. 6 (1982), 409–416.

[61] J. Šeděnková: On the invariant variational sequences in mechanics. In: Rend. Cont. Mat. Palermo. Proc. 22nd Winter School Geometry and Physics. Srní, Czech Republic.

[62] J. Štefánek: A representation of the variational sequence in higher order me- chanics, in: *Differential Geometry and its Applications*, Proc. Conf., Brno, Czechoslovakia, September 1995, J. Janyška, I. Kolář, J. Slovák, eds.: Masaryk University, Brno, 1996, 469–478.

[63] F. Takens: A global version of the inverse problem of calculus of variations, J. Differential Geometry 14 (1979), 543–562.

[64] W. M. Tulczyjew: The EulerLagrange resolution, in: *Internat. Coll. on Diff. Geom. Methods in Math. Phys.*, Aix-en-Provence, 1979; Lecture Notes in Math. 836, Springer, Berlin, 1980, 22–48.

[65] A. M. Vinogradov: A spectral sequence associated with a non-linear differential equation, and the algebro-geometric foundations of Lagrangian field theory with constraints, Soviet Math. Dokl. 19 (1978), 790–794.

[66] A. M. Vinogradov, I. S. Krasilschik, V. V. Lychagin: *Introduction to the Geometry of Non-linear Differential Equations* (Russian), Nauka, Moscow, 1986.

[67] R. Vitolo: Finite order Lagrangian Bicomplexes, Math. Proc. Cambridge Phil. Soc. 125 (1999).

[68] R. Vitolo: On different geometric formulations of lagrangian formalism, Diff. Geom. Appl. 10 (1999), 225–255.

Index