Chronology violation, charged perfect fluid and exact solutions of Einstein's equations

Pavel Klepáč

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Masaryk University
Faculty of Science
Institute of Theoretical physics and Astrophysics
Kotlářská 2, 611 37 Brno

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Contents

Notation

M	manifold
S^n	n–dimensional sphere
$\times$, $\otimes$	direct and tensor product
$p \prec q$	p chronologically precedes q
$p \ll q$	p causally precedes q
$I^+(p)$, $I^-(p)$	chronological future and past of p respectively
$J^+(p)$, $J^-(p)$	causal future and past of p respectively
$\overline{U}$, ∂U	closure and boundary of the set U respectively
$\emptyset$	empty set
$D^+(S)$, $D^-(S)$	future and past Cauchy development of the set S respectively
$H^+(S)$, $H^-(S)$	future and past Cauchy horison of the set S respectively
$\alpha, \beta, \ldots$	Greek indices range over 0,1,2,3
$\hat{\alpha}, \hat{\beta}, \ldots$	hatted indices mean an orthonormal basis
$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$	basis vectors of a coordinate system
$\mathrm{d}x^\mu$	basis 1–forms of a coordinate system
$\mathbf{e}_\mu$	basis vectors of an arbitrary basis
Θ^μ	basis 1–forms of an arbitrary basis
g	metric tensor, determinant of the metric tensor
η	volume element
$\epsilon_{i_1 i_2 \ldots i_3}$	Levi–Civita permutation symbol
$*$	Hodge dual
$\wedge$, d	exterior product and exterior derivative
p	spacetime point, pressure
μ	energy density
ρ	mass density, charge density
j	flow vector
$\mathrm{E}_{\hat{i}}$, $\mathrm{B}_{\hat{i}}$	components of electric and magnetic field
ω	vorticity 1–form
F	electromagnetic field 2–form
${\omega^\alpha}_\beta$, ${\Omega^\alpha}_\beta$	connection 1–form and curvature 2–form respectively
$R_{\alpha\beta\gamma\delta}$, $R_{\alpha\beta}$, R	components of Riemann and Ricci tensor, Ricci curvature
R^2_{GB}	Gauss–Bonnet term

Chapter 1

Introduction

A physical theory which aims to be a consistent model for the observable physical phenomena should at the first place successfully explain already known physical laws, but it should also predict new, yet unrecognised features and phenomena. So far we are aware of four fundamental interactions in the nature – weak, strong, electromagnetic and gravitational. Among them, only the gravitational interaction is universal, and every physical event is either directly or indirectly influenced by it.

History of science has noticed various theories of gravity, but only few have overcome the crucial criteria required for a correct physical theory. Between them, the Einstein's relativity is widely believed to be most consistent classical gravity theory because of its logical simplicity, geometrical elegance and physical transparency. The accuracy of the general relativity has been many times confirmed by most precise experimental equipment.

The study of exact solutions of the Einstein field equations surely belongs to the physical areas of the greatest importance [1, 2, 3, 4].

However, although Einstein's relativity is an excellent approximation of the gravitational effects occurring at low energies, as one approaches to the Planck energy scales, one has to incorporate quantum corrections. The relationship between micro and macrophysics is most usually sought within a framework of unified description of the fundamental forces, that are proved by string and superstring theories [5].

Many serious questions arise when one passes from general relativity to string theory. One of them which concerns us in this thesis is the following. Relativity theory is known to posses the strange property that there exist solutions to Einstein's equation in which travelling into the coordinate past is possible. The Gödel-type solutions [6, 7] are well known examples. Is there any hope that the higher-order stringy corrections occurring in string theory would prevent the closed timelike curves (CTC's) to being formed?

The main subject of this PhD thesis is looking for exact solutions of the Einstein's field equations, or some of their generalizations. If such a search has been successful then an attempt is made to examine some basic properties of the solutions found. Among them the chronological structure, occurrence of singularities, geometry of congruences and energy conditions are especially interesting. The spacetimes dealt with are cylindrically symmetric and stationary, and moreover they belong to the so called rotating universes.

Let us explain the reasons and motivations for doing this. In 1949 Kurt Gödel published paper [6] in which he gave new solution of Einstein's field equations describing a homogeneous rotating universe filled with a dust (and non-zero cosmological constant). It turned out that it had many interesting geometrical features. In particular, it contained closed timelike curves. Gödel spacetime had become a gold mine for many scientists and let to the serious examination of global spacetime structure.

This thesis is divided into two parts. First of them has been intended to explain some technical and conceptual points connected with the topic discussed, as well as to provide an introduction to the papers included in second part.

Chapter 2 have been included since it provides the theoretical background for causal structure concepts studied further. Besides the basic concepts, the proofs of a number more or less difficult statements in which I was involved are given. The chapter is self–contained, but the results are not in any sense original.

The structure of chapter 3 is the following. In section 3.1 we mention the Gödel-type spacetimes. Section 3.2 provides a guide across the obtained results in the classical relativity. There are given and outlined some results. The section is not in any case complete, the detailed explanation is contained in first two papers of second part [8, 9]. Section 3.3.2 shows an economical way for the field equations to be derived. In section 3.4 we analyse the character of the chronology violation in greater detail, both theoretically and practically.

Finally section 4 is intended to drawn the attention to the overlap between the general relativity and string theory. After an introduction of the Gauss–Bonnet theorem in section 4.1 and reasons for an incorporation of the higher–order derivative terms in the curvature into the ordinary general relativistic action, discussed in section 4.2, we very briefly outline the obtained results and comment on possible directions of next considerations in section 4.3. Again, for obtained results the reader is referred to the remaining two papers [10, 11].

Not every aspect of the topics treated in the publications included is mentioned in the first part, namely the energy conditions, geometry of congruences and the singularities, but these topics can be found in standard textbooks [12, 3], so the author omitted them in the thesis. References are given at the end of every chapter.

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Chapter 2

Chronological structure – review

2.1 Spacetime manifolds, orientability

A *spacetime* is a connected, paracompact and Haussdorf manifold carrying a Lorentzian metric[1]. It turns out that the existence of Lorentzian metric on a given manifold is equivalent to the existence of the *directional field.* The directional field is everywhere non–vanishing vector field, at each point defined up to an arbitrary multiple. It can be shown [1, 2] that every non–compact manifold may be endowed with Lorentzian metric. If the compact manifolds are considered the situation is more complicated and the necessary and sufficient condition for a given compact manifold to carry the Lorentzian metric is that its Euler characteristic[2] vanishes.

There are two powerful methods in the spacetime global structure study.

The first of them is the "cutting of holes". Let M be a manifold and D its closed subset. Then $M - D$ is manifold with metric g induced from M, which arises from M by removing the points of D. Clearly, the condition for D to be closed is necessary for $M - D$ to be even a manifold.

As example let us mention the Minkowski spacetime with an origin, a line or a plane removed. The resulting topology is $S^3 \times \mathbb{R}$, $S^2 \times \mathbb{R}^2$ and $S^1 \times \mathbb{R}^3$, respectively.

The second method is "patching together". For example let us consider a manifold with boundary formed by two diffeomorfic 3–dimensional manifolds Σ_1 and Σ_2. These two manifolds can be identified in order to obtain 4–dimensional manifold (identification means a diffeomorfism from Σ_1 to Σ_2).

A spacetime is called *time orientable* if at each its point the future and the past directed timelike vectors can be chosen so that the choice is continuous [3]. If the spacetime is time orientable, there exist just two *time orientations.* One can analogically define also *spatial orientation.*

For a given spacetime (M, g) let us consider a fixed time orientation at p, and a loop γ passing through p. Now one can carry the orientation along γ back to p. The resulting

[1] Throughout the thesis we use the signature $(+, -, -, -)$.

[2] For the definition of Euler characteristic see section 4.1.

orientation will be either the same or opposite to the original one. We say that γ is either time orientation preserving or reversing. It is easily seen that (M, g) is time orientable if and only if each loop is time orientation preserving.

Finally, a curve is *spacetime orientation* preserving if it either preserves both time and space orientation or reverses both of them. Note that the concept of the spacetime orientation is metric field independent. Familiar example in two dimensions is the Möbius band.

Because on a simply connected manifold each curve can be continuously shrinked to some point, and since one can always choose time orientation continuously in a sufficiently small neighborhood of any point, it follows that every simply connected spacetime is time orientable.

2.2 Causal structure

Henceforth we will assume that spacetime considered is time orientable. There is no loss of generality since even if an original spacetime is not time orientable, its universal covering spacetime, being simply connected by definition, already is.

The topology gives us only poor information about physical structure of the spacetime. Fortunately, there is a certain structure which is in fact a transition between the topology and the metric. It appears when one analyses the fundamental questions like whether certain event could influence another one, or whether the evolution of a given event could be uniquely predestinated by a physical ground imposed on certain set of events.

We say that a point p *chronologically precedes* a point q if there exists future directed timelike curve from p to q. This relation is written as $p \ll q$. By timelike curve we mean a curve with everywhere positive norm of the tangent vector.

Analogically, a point p *causally precedes* a point q if there is future directed causal curve from p to q. This is written like $p \prec q$. By causal curve is meant a curve whose tangent vector is nowhere negative, so in particular a curve that always remains in p is causal.

Chronological future $I^+(p)$ of the point p is the set of all points q such that $p \ll q$. Similarly, *chronological past* $I^-(p)$ of the point p is the set of all points q such that $q \ll p$.

Causal future $J^+(p)$ and *causal past* $J^-(p)$ of the point p are defined in the same manner simply by replacing the word timelike by the word causal in the above definitions of $I^+(p)$ and $I^-(p)$.

For example, if we take the Minkowski spacetime, and p is the origin, then $I^+(p)$ is future light cone interior, $I^+(p) = \{(t, x, y, z) \in M : t > (x^2 + y^2 + z^2)^{1/2}\}$. Causal future of p is formed by the future light cone plus its interior.

A very basic property of the chronological pasts or futures is that they are always open [4]. Let us consider for example $I^+(p)$ of a point p. We must show that for any $q \in I^+(p)$, $I^+(p)$ contains also a small neighborhood of q. But this follows immediately, since one can slightly deform a timelike curve from p to q such that it remains timelike. For, fix a point $r \in I^+(p)$, $r \ll q$, belonging to a sufficiently small neighborhood U of q. Then q chronologically precedes any point from a neighborhood V of q, $V \subset U$. Because

Figure 2.1: Smoothing out corners at q.

$V \subset I^+(r)$, the set $I^+(p)$ contains V.

Note that causal futures need not be either open or closed. For example Minkowski consider spacetime with a point removed.

If $p \ll q$, then $I^+(q) \subset I^+(p)$. For, note that if $p \ll q$ and $q \ll r$, then $p \ll r$. Indeed, one can compose a future directed timelike curve from p to q with a future directed timelike curve from q to r and smooth out the corner (if any) at q, see figure 2.1. The resulting curve will be timelike (although need not necessarily contain q). Now for any $r \in I^+(q)$ is $q \ll r$, and thus $r \in I^+(p)$. On the other hand, the converse is false, for example take origin in Minkowski spacetime and the point $(1, 1, 0, 0)$.

The following implications are also important. If it holds $p \ll q \prec r$, or $p \prec q \ll r$, for points p , q and r, then $p \ll r$ [4]. The proof uses the basic property of the causal curves: if a causal curve, which is not null geodesic, joins two points, then always exists its variation yielding a timelike curve between these points [1, 4]. Now, compose the timelike curve from p to q with the causal curve from q to r. Such composition is certainly causal curve, but it can not be null geodesic. Therefore it can be deformed into a timelike curve from p to r.

It can be straightforwardly [1] shown that the boundaries as well as the closures of the chronological and causal futures are the same

$$\partial I^+(p) = \partial J^+(p) \ , \qquad \overline{I^+(p)} = \overline{J^+(p)} \ .$$

For, let us suppose $q \in \overline{J^+(p)}$. Consider a point r such that $q \ll r$. Because $I^-(r)$ is open, there exists a neighborhood U of q for which $U \subset I^-(r)$. But every such U intersects $J^+(p)$. So there is a point $r' \in U \cap I^-(r)$, for which $p \prec r' \ll r$. According to the previous paragraph $r \in J^+(p)$. If, on the other hand, q is in $\overline{I^+(p)}$, then every neighborhood contains the points from $I^+(p) \subset J^+(p)$. From this it also follows that a future directed causal curve passing through p and nowhere intersecting $I^+(p)$ is null geodesic.

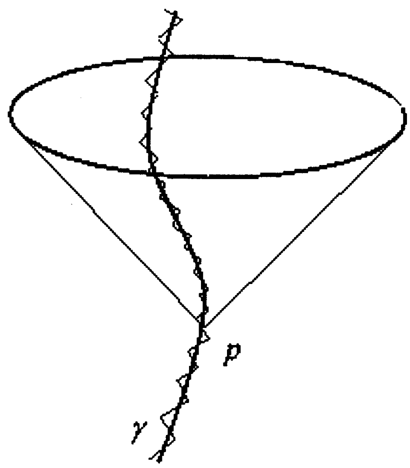

Figure 2.2: Approximation of a timelike curve by the null one.

Clearly, the set $I^+(p)$ is the union of all sets of the form $I^+(q)$, where $q \in I^+(p)$.

Let us briefly comment on some further properties of causal structure. If $p \prec q$ then p and q can be joined by future directed null curve. It follows from the fact that an arbitrary timelike curve γ can be approximated, for example by means of mirror system, by some null curve lying in neighborhood of γ, see figure 2.2.

There are many truly sounding statements on the causal structure field, which are in fact false. For example it holds that the relation $p \ll q$ implies $I^+(q) \subset I^+(p)$. But converse fails for example for $q = (1, 1, 0, 0)$ and p being the origin in ordinary Minkowski spacetime.

Even if $I^+(q) \subset I^+(p)$, it may hold $I^-(p) \not\subset I^-(q)$. For, take two points $p = (-1, -1, 0, 0)$ and $q = (1, 1, 0, 0)$ in Minkowski spacetime with the semi-plane $t = 0, x \geq 0$ removed as is shown on the figure 2.3. From the equality $I^+(p) = I^-(q)$ and $I^-(p) = I^-(q)$ it need not follow $p = q$. This is demonstrated by the example of cylindrical universe obtained from Minkowski spacetime by taking the strip $|t| \leq 1$ and identifying the points $(-1, x, y, z)$ with points $(1, x, y, z)$. Suppressing two dimension we have 2-dimensional cylindrical universe on the figure 2.4. Even if the chronological futures of two points p and q do not intersect, their pasts can. It holds for example for $p = (0, -2, 0, 0)$ and $q = (0, 2, 0, 0)$ in Minkowski spacetime with disk $x^2 + y^2 + z^2 \leq 9, t = 1$ removed, see figure 2.5.

Non-empty set H equal to its future and past, is the entire spacetime. This is seen as follows. H is open. Suppose that H does not coincide with M and consider its complement in M, which is closed. Then a point q with the following properties must exist: q does not lie in H, but every its neighborhood has non-empty intersection with H. One can find a

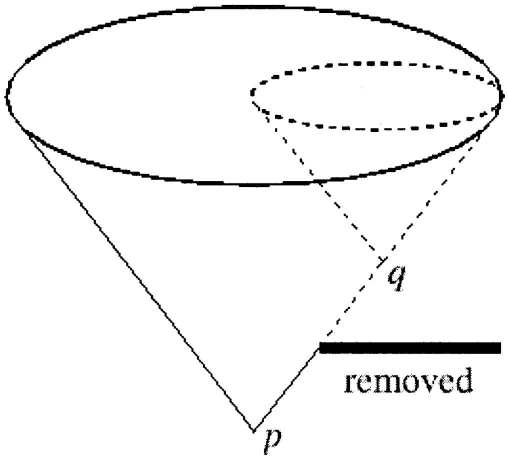

Figure 2.3: The relation $I^+(q) \subset I^+(p)$ does not imply $I^-(p) \not\subset I^-(q)$.

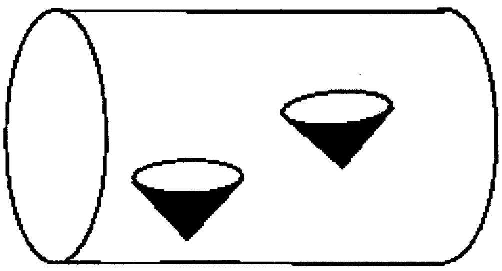

Figure 2.4: The relation $I^+(q) \subset I^+(p)$ does not imply $I^-(p) \not\subset I^-(q)$.

timelike curve joining q to H, which means that $q \in H$. This is contradiction. Let p be the fixed spacetime point. The point p is called to have index n if the future of the past of the future of the ... of the future (n times) of p is the whole spacetime although for $(n-1)$ it is not (this sequence always begins with the future of p). By the construction one finds that there are spacetimes not having finite index. An example obtained from the Minkowski spacetime by removing two hyperplanes is depicted on the upper figure 2.6 with two dimensions suppressed. The lower figure 2.6 shows that for each integer there exists a spacetime with the index equal to this integer. In the latter case two discs with the radii $(n-2)$ and $(n-1)$ are suitably removed from the Minkowski spacetime.

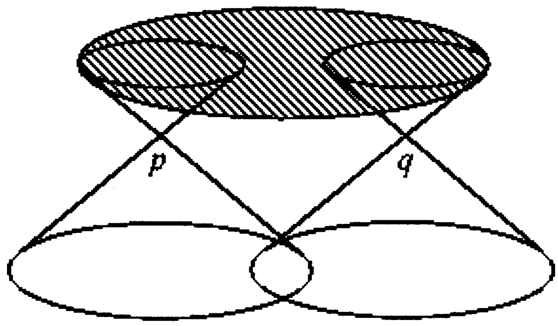

Figure 2.5: Empty intersection of futures of two points with non–empty intersection of their pasts.

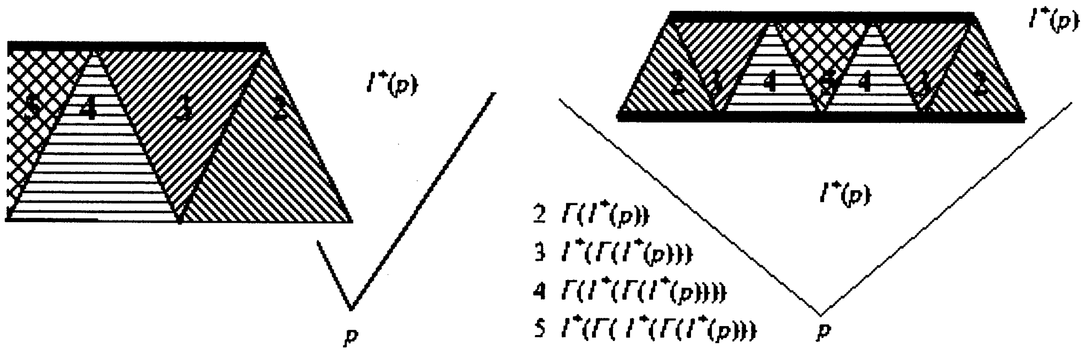

Figure 2.6: The upper figure shows the spacetime with infinite index. The successively reached levels of the futures and pasts are marked by different hatching. In the lower figure, there is the example of the spacetime with any integer as the index for $n = 5$.

2.2.1 Chronology conditions

The time travel possibility has excited many authors for past two centuries and surely will fascinate them in future. In fact this problem is rather delicate and its discussion from both physical and philosofic points of view is so wide that even an outline how it can be solved in terms of modern physics is well behind the scope of this work. Here we will restrict ourselves merely to the introduction of chronology conditions.

Obviously, one would like the time travel not to exist, from a number of reasons, at least

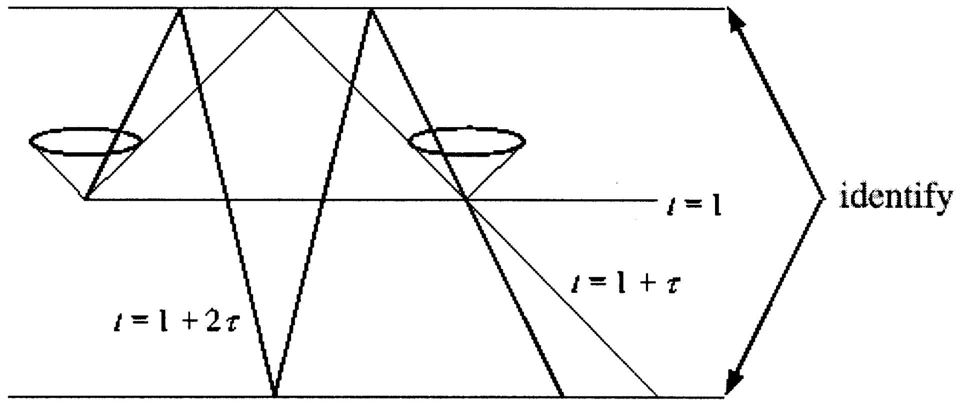

Figure 2.7: The spacetime with two points joined simultaneously by timelike, null and spacelike geodesics.

on the field of physical considerations. *Chronology violating region* $I^0(M)$ of a spacetime is the set of all points q such that there exists closed timelike curve passing through q. A spacetime satisfies *chronology condition* if for no its point p it holds $p \ll p$. In other words, the chronology violating region is empty and there are no closed timelike curves present (CTCs) in the spacetime.

One of the simplest spacetimes with CTCs is obtained by removing two open 4–dimensional balls centred at the point $(2, 0, 0, 0)$ and at the origin, with radii smaller than 1, from Minkowski spacetime. Now we identify the 3–spheres S^3, boundary of these balls, symmetrically with respect to the $t = 1$ hyperplane. Essentially this example describes the time machine built on the wormhole [5]. Every spacetime (M, g) built on compact manifold M violates the chronology condition [1].

If a spacetime contains at least one CTC, then it has infinitely many CTCs. Indeed, let γ be CTC and choose two points p and q on γ, $p \ll q$. The sets $I^+(p)$ and $I^-(q)$ are open and so is their intersection U. For any $r \in U$ join a future directed timelike segment from p to r to a future directed timelike segment from r to q and smooth out the corner (if any) at r.

There are spacetimes with the points p and q such that they can be joined simultaneously by timelike, null and spacelike curve. The example is shown on the figure 2.7 with two dimension suppressed. It is well known cylindrical universe – strip of the Minkowski spacetime between the hypersurfaces $t = 0$ and $t = 2$ identified. Take the point $(1, 0, 0, 0)$ and consider family of geodesics $t = 1 + \varepsilon\tau, x = \tau, y = z = 0$, with τ being affine parameter, and ε real number. Choose $\varepsilon = 2, 1, 0$ for the timelike, null and spacelike geodesics respectively.

Even if the chronology condition is guaranteed, other pathological features still oc-

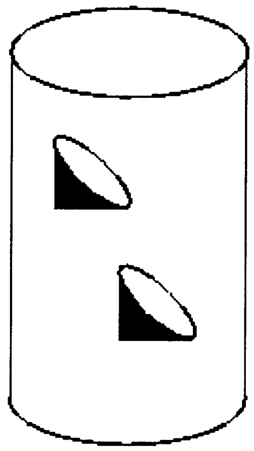

Figure 2.8: The spacetime with closed null curves but without CTCs.

cur. For instance, there are spacetimes without CTCs but in which closed null curves are present. For, consider strip of the Minkowski spacetime between the hypersurfaces $t = 0$ and $t = 1$, and identify the points $(0, x, y, z)$ and $(1, 1 + x, y, z)$. Suppressing two dimensions, this spacetime can be imagined either as the cylinder with induced metric $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2$, rotated by angle $\frac{\pi}{4}$, or as an ordinary (standing) cylinder with induced metric $\mathrm{d}s^2 = \mathrm{d}t\ \mathrm{d}\varphi\ ,\ \ \varphi \in [0, 2\pi)$, see the figure 2.8.

Note that in the previous example the closed null curves were actually geodesics. This is general phenomena, because every null curve, which is not geodesic, can be deformed into a timelike curve. Thus if a given spacetime satisfies the chronology condition, but violates the causality condition, the only closed causal curves are null geodesics.

Causality violating region $J^0(M)$ is the set of all points through which a closed causal curve passes. A spacetime satisfies *causality condition* if it does not contain any closed causal curve. It can be reformulated in terms of non–existence of any two distinct points p and q such that $p \prec q \prec p$.[3]

Future chronology horizon of the set S is defined as $\partial I^+(I^0(M))$ and *future causality horizon* by $\partial J^+(J^0(M))$.

So far we have introduced two most basic conditions imposed on the chronological structure of the spacetime. However there are further stronger conditions that are physically reasonable. This is because what one really wants is an exclusion of nearly closed causal curves, i.e. curves which return infinitesimally close to some of their points when they already left a sufficiently small neighborhood of this point. It motivates the following considerations.

[3]The definition requires two distinct points since $p \prec p$ for any point identically.

Since the sets of the type $I^-(q) \cap I^+(p)$ are always open, we may ask whether they could serve as the topological basis of the manifold M. In other words, we may ask if every open set could be expressed as an union of the sets of this type. But the cylindrical universe shows that it is not generally possible. In this example $I^-(p) = M$ for any point p, and so the sets $I^-(q) \cap I^+(p)$ are too large for an arbitrary set to be expressible with the help of them.

It turns out that the sets of the type $I^-(q) \cap I^+(p)$ may serve as the topology basis if a spacetime satisfies *strong causality condition.* A spacetime is strongly causal at a point p if for each neighborhood of p exists a neighborhood of this point, such that no causal curve enters it more than once. (M, g) is strongly causal if it is strongly causal at each point. Then it follows that the set $I^-(q) \cap I^-(p)$ may be chosen as the topology basis.[4]

We briefly mention the concept of a *captured* curve. A future directed, future inextendible, causal curve may

- enter and remain in a compact set – completely captured curve
- eternally leave and return to a compact set – partially captured curve
- reach the "edge" of the spacetime [5] – the curve leaves and returns to a compact set only finite number times

It can be seen that if the strong causality is fullfiled in some compact set H, then there are no future inextendible causal curves which are partially captured in H in the future. This is because H can be covered by finite number of open sets. If any future directed causal curve enters H infinitely times a compact set with a finite covering, it has to enter at least one of the covering sets infinitely times, thus violating the strong causality.

Very important is the concept of *achronal set.* It is a set S such that no two its points can be joined by a chronological curve, or more precisely $I^+(S) \cap S = \emptyset$.[6] For example the light cone of a point p is achronal set.

In some works [1] the concept of *future set* B is introduced, for which its chronological future is subset of B, $I^+(B) \subset B$. For example the sets $I^+(S)$, $J^+(S)$ for any set S are all future sets.

A future set is in fact achronal. It follows from the fact that for any point $p \in \partial B$ it holds $I^+(p) \subset B$, $I^-(p) \subset M - B$. The proof runs as follows. For $p \in \partial B$ consider a point $q \in I^+(p)$. Since $I^-(q)$ is non–empty, it contains some neighborhood U of the point p. But every neigborhood intersects B. If $q \notin B$, there would exist future directed timelike curve from a point $r \in U \cap B$ to q, which would contradict the fact that $I^+(r) \subset B$. On the other hand, if there was $r \in I^-(p)$ and lying in B, there would be a neighborhood U' of p entirely lying in $I^+(r)$. And because every such neighborhood has non–empty intersection with $M - B$, there would exist future directed timelike curve from r to $M - B$, which

[4] The topology in which a set is open if and only if it can be obtained as an union of the set $I^-(q) \cap I^-(p)$ is called Alexandrov topology.

[5] By edge we mean either a singularity or an infinity of the spacetime.

[6] By $I^+(S)$ we mean the collection $\cup_{p \in S} I^+(p)$.

is the contradiction. Since an arbitrary $q \in I^+(p)$ belongs to B, we can always find a neighborhood V of the point q which is contained in B. Thus q is the inner point of B, $\partial B \cap I^+(\partial B) = \emptyset$.

There exist spacetimes with no achronal set. For example in the Gödel spacetime, which will be treated in next sections, every two points can be joined by chronological curve, thus there are no achronal sets.

The reason for imposing next condition, even stronger then the previous ones, relies on the following [1]. General relativity is a classical theory, but our real world must be driven by quantum laws as well. But the quantum mechanics laws rule out the possibility that the metric has at all spacetime points exactly defined value due to uncertainty principle. As a result, we must exclude not only CTCs in spacetime (M, g) , but also in the spacetime $(M, \tilde{g})$, where $\tilde{g}$ is a metric close (in sense defined bellow[7]) to the original metric g.

A spacetime (M, g) is *stably causal* if there exists everywhere non–vanishing vector field t such that the Lorentzian metric on M, given as $g + t \otimes t$ does not admit any CTCs.

The light cones corresponding to $g + t \otimes t$ are "wider" than the ones corresponding to g. So, if a spacetime violates any of the previous condition, it will necessarily violate the stable causality too.

Time function f is scalar field $f : M \to \mathbb{R}$ defined over the whole M, which is increasing along any future directed causal curve. If a spacetime admits the time function, it is stably causal. It follows from the fact that in the previous definition we can choose $t = \nabla f$.

The *future distinguishing condition* at a point p is fulfilled if for each neighborhood of p there exists a neighborhood of p such that no future directed causal curve intersects it more than once. A spacetime satisfies future distinguishing condition if it satisfies this condition at each point. Equivalently[8], from the relation $I^+(p) = I^+(q)$ it follows $p = q$ for any p and q.

A spacetime is *causally simple* if for any compact subset K, both $J^+(K)$ and $J^-(K)$ are closed.

Finally, a spacetime is *globally hyperbolic* if and only if there exists the so called Cauchy surface, which will be treated in next section.

The hierarchy of these conditions is shown on the figure 2.9.

2.3 Cauchy development

Viewpoint of determinism was widely spread in eighteenth and nineteenth centuries, and it was accepted in particular by the creators of the analytical mechanics like Laplace, Lagrange and Hamilton. The idea of determinism is such that the state of the universe at any instant uniquely predestinates the behaviour in the whole future of the universe. The problem of determining the time evolution of a dynamical system restricted by some initial conditions, is studied in the theory of differential equations in an exhausting manner. One of the first scientists who successfully tried to cope with problem was Louis Cauchy,

[7] See also [1] for the definition of the topological details.

[8] Similarly *past distinguishing condition* can be defined by replacing "future" with "past".

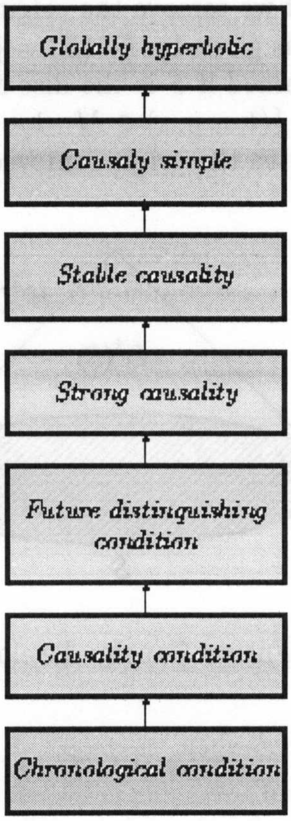

Figure 2.9: Hierarchy of the conditions imposed on the chronological structure. The arrows shows the direction of decreasing generality.

in honour of whom it was named. The discovery of general relativity and examination of its consequences in the field of exact solution to the Einsteins equations have shown, that the determinism, in the sense just outlined, is incompatible with the theory. By evolution we mean the behaviour of the gravitational and possibly others fields present in a spacetime. In section 2 concerning causal structure, we have shown several examples with CTCs. Intuitively, the set of all points at which evolution of the physical fields is uniquely determined by the initial conditions imposed on a given hypersurface, could be defined as a spacetime region without holes or CTCs. In other words we wish to exclude the possibility of an additional information entering from elsewhere which could, in principle, destroy the information coming from the hypersurface.

Future Cauchy development $D^+(S)$ of the spacetime subset S is defined as the set of all points such that every past directed inextendible timelike curve intersects S. Here we assume that $D^+(S) \supset S$.

Clearly, if S is the hyperplane $t = 0$ in the Minkowski spacetime, then $D^+(S) = \{(t,x,y,z) \in M : t \geq 0\}$. But if we remove the origin from the spacetime manifold, then the future Cauchy development is given by $D^+(S) = \{(t,x,y,z) \in M : (x^2+y^2+z^2)^{1/2} \geq t \geq 0\}$. Another example is obtained if S is the disc $S = \{(t,x,y,z) \in M : 1 \geq x^2+y^2+z^2, t = 0\}$. In this case $D^+(S) = \{(t,x,y,z) \in M : 1-(x^2+y^2+z^2)^{1/2} \geq t \geq 0\}$ (see figure 2.10.) $D^+(S)$ of the same disc with the origin removed is depicted on the figure 2.11.

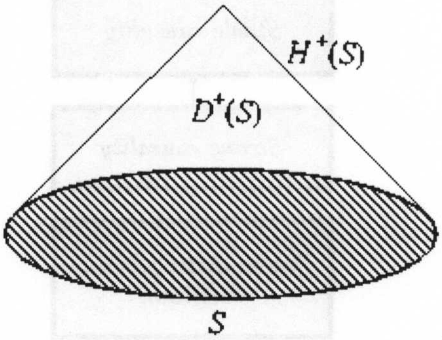

Figure 2.10: Future Cauchy development of a disc.

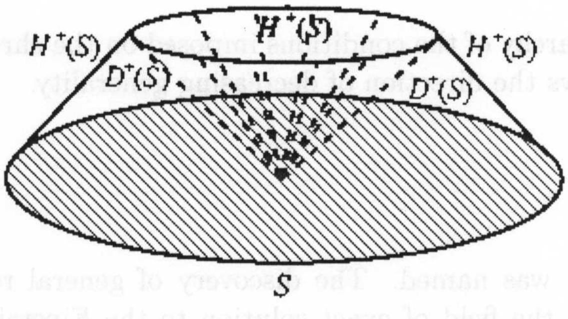

Figure 2.11: Future Cauchy development of a disc with the origin removed.

Hereafter we will assume that the set S in the definition of $D^+(S)$ is achronal. It corresponds to the fact that typically the initial conditions are imposed on a spacelike hypersurface.

Furthermore we mention some elementary properties of the Cauchy development.

Both from its definition and the above examples it is seen that not every inextendible past directed causal curve passing through a point in $D^-(S)$ intersects S. It occurs when such point lies on the boundary of $D^+(S)$ (the concept of boundaries for Cauchy developments will be specified later). Indeed, if $r \in D^+(S)$, then every inextendible past directed curve intersects S, because there is a point $p \in I^+(r)$ and involved in $D^+(S)$. If there was a curve λ passing through r which does not intersect S, there would be past directed timelike curve from p lying entirely in the future of λ, and thus not intersecting S.

No point from $D^+(S)$ cannot lie on a CTC. For, if $p \in D^+(S)$ lies on CTC γ that does intersect S, then S cannot be achronal (there would exist a point $r \in S$ such that $I^+(r) \cap S \neq \emptyset$). If, on the other hand, γ did not intersect S, the point p could not belong to $I^+(S)$.

If $p \in D^+(S)$ and $q \ll p$, then $q \in D^+(S)$. For, if $q \notin D^+(S)$, then there would exist causal past directed curve γ from q not intersecting S. Let us join γ to past directed timelike curve from p to q, and smooth out a possible corner at q. The result is timelike curve not intersecting S, which is contradiction. From this it also follows that

$$I^-(p) \cap I^+(S) \subset D^+(S) \quad \text{for } p \in D^+(S) .$$

The above examples with the hyperplane show that generally

$$D^+(\bar{S}) \neq \overline{D^+(S)} .$$

Replacing "future" and "past" in the previous paragraphs one can define *past Cauchy development* $D^-(S)$ of the set S.

It can be easily seen that if any point p belongs both to $D^+(S)$ and $D^-(S)$, then it must lie on S. For, if p is not contained in S, there would exist future directed timelike curve γ from p intersecting S, and past directed timelike curve $\tilde{\gamma}$ from p, also intersecting S. Then the composition (possibly after smoothing out the corner at p) gives timelike curve from S to S, which is the contradiction.

Let us return to the case in which S was the disc $S = \{(t, x, y, z) \in M : 1 \geq x^2 + y^2 + z^2, t = 0\}$. The point is that if S' was the disc $S' = \{(t, x, y, z) \in M : 1 > x^2+y^2+z^2, t = 0\}$, i.e. S' would differ from S by the sphere C given by $x^2 + y^2 + z^2 = 1$, then $D^+(S)$ would differ from $D^+(\tilde{S})$ by C, $D^+(\tilde{S}) = D^+(S) - C$. This result is generalized in the following way [6]

$$\overline{D^+(S)} = D^+(S) \cup \bar{S} .$$

In particular, if S is closed, then $D^+(S)$ is also closed.

Cauchy development $D(S)$ of the set S is the union $D(S) = D^+(S) \cup D^-(S)$. If it happens that $D(S) = M$, the achronal set S is called *Cauchy surface.* However, the existence of the Cauchy surface in a spacetime is a very strong restriction, and most of the known general relativistic spacetimes do not admit any Cauchy surface. For example Minkowski and Schwarzchild solutions belong to globally hyperbolic spacetimes (i.e. those admitting Cauchy surface). But it is sufficient to remove origin from the Minkowski spacetime, and the resulting manifold does not admit any Cauchy surface, because we can always find an inextendible timelike curve ending in the removed origin). From further important examples, the family of Kerr–Newmann solutions and its special case – Reissner–Nordström solutions – in general is not globally hyperbolic. Very important is the concept of the Cauchy developments boundaries [1, 6]. The set of the points $p \in D^+(S)$ such that none of them chronologically precedes any other point in $D^+(S)$ is called *future Cauchy horizon* $H^+(S)$,

$$H^+(S) = D^+(S) - I^-(D^+(S)) .$$

Future Cauchy horizon is essentially the boundary between causally well behaved region $D^+(S)$ and a region where the evolution can not be uniquely determinated. For example $H^+(S)$ of the disc from the figure 2.10 is constituted by envelope of the turned cone. How $H^+(S)$ looks like for the disc without the origin is shown on the figure 2.11. Note that $H^+(S)$ intersects S (and at the same time $D^+(S)$ is closed). Nevertheless this is not general result what is seen immediately if we take the same disc but without its boundary C (now $D^+(S)$ is not closed). In general S, $D^+(S)$ and $H^+(S)$ are either all simultaneously closed or none of them is [6].

Directly from the definition it is clear that no two points lying on $H^+(S)$ can not be joined by timelike curve, and thus $H^+(S)$ is achronal. One also obtains that $D^+(H^+(S)) = H^+(H^+(S)) = H^+(S)$.

Past Cauchy horizon $H^-(S)$ and the (total) *Cauchy horizon* are defined similarly as the past and (total) Cauchy development.

The following result holds. Any two Cauchy surfaces S_1 and S_2 in a given spacetime share the same topology. The idea of the proof consists of a construction of an everywhere defined timelike vector field ξ. In this way every point $p \in S_1$ is mapped onto $q \in S_2$, where q is the intersection of the integral curve of ξ passing though p with S_2. The existence of such curve follows from the fact that S_2 is Cauchy surface. This mapping is continuous and because S_1 is achronal, two distinct points on S_1 can not have the same image. Thus it is bijection. Reversing roles of S_1 and S_2 yields that the mapping is in fact homeomorphism.

We finish this section with the following statement. The necessary and sufficient condition for an achronal set S to be a Cauchy surface is that the $H(S)$ is an empty set[9]. Suppose that S is the Cauchy surface. Then $D(S) = M$ and $H(S)$ must be empty. On the other hand, if $H(S) = \emptyset$, for each point $p \in D(S)$ there must exist future directed timelike curve having non–empty intersection with $D(S)$. Let $r \in D(S)$. But since $H^-(S) = \emptyset$, there is $q \in D(S)$ such that q can be joined to p by future directed timelike curve. Now, for sufficiently small neighborhood U of p it holds $U \subset I^+(q) \cap I^-(r) \subset D(S)$. Thus $D(S)$ is non–empty and open. But $D(S)$ is simultaneously closed since $H(S) = \emptyset$ is closed. But the only set which is simultaneously both closed and open, is the spacetime alone.

[9] Here we assume that M is connected.

References for chapter 2

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Chapter 3

Cylindrically symmetric spacetimes

3.1 Gödel–type solutions

The Gödel solution [1] is a special case of more general Gödel-type metric in the cylindrical coordinates defined by the line element

$$ds^2 = [dt + H(r)d\varphi]^2 - D^2(r)d\varphi^2 - dz^2 - dr^2 \ , \tag{3.1}$$

Clearly, (3.1) describes a stationary spacetime.

Raychaudhuri and Guha Thakurta [2] found the necessary condition for the Gödel–type spacetime to be (spacetime) homogeneous. Later on Rebouças and Tiomno [3] have shown that the condition is also sufficient. Thus according to [3] the Gödel–type spacetime is homogeneous if and only if the following two equations are satisfied

$$\frac{D''}{D} = \text{const} \equiv m^2 \ , \qquad \text{and} \qquad \frac{H'}{D} = \text{const} \equiv 2\Omega \tag{3.2}$$

This means that metric (3.1) of the Gödel–type homogeneous spacetime can take either one of the three forms, depending on the value of the constant m. The corresponding functions D and H are

a) $H = \Omega r^2, \ D = r \quad \text{if } m = 0 \ ,$

b) $H = \frac{4\Omega}{m^2}\text{sh}^2\left(\frac{mr}{2}\right), \ D = \frac{1}{m^2}\text{sh}\,(mr) \quad \text{if } m^2 = \text{const} > 0 \ ,$

c) $H = \frac{4\Omega}{\beta^2}\sin^2\left(\frac{\beta r}{2}\right), \ D = \frac{1}{\beta^2}\sin\,(\beta r) \quad \text{if } \beta^2 = -m^2 > 0 \ ,$

Two solutions denoted by the same letter are equivalent if and only if they share the same Ω and m. Solutions denoted by different letters are non-equivalent. The physical meaning of the parameter Ω is that of vorticity scalar.

Since the metric (3.1) is homogeneous, it has at least four Killing vectors. Because it is cylindrically symmetric, having fifth Killing vector ∂_φ, the five Killing vectors form Lie algebra, which is generally five–dimensional. In fact there is closed subalgebra of the Lie algebra, isomorphic to $so(2,1)$ (see [4] and references therein). Particular cases of (3.1) have been found with six and seven–dimensional Lie algebras[1] [6].

Topologically the Gödel–type spacetime is a product of a curved three–dimensional manifold with the real line in ∂_z direction. This implies [5] the existence of the only affine collineation $\xi = (az + b)\partial_z$ with constants a and b.

The original Gödel solution [1] emerges if the perfect fluid is taken as the matter content of the spacetime. The relationship between fundamental parameters Ω and m then becomes $2\Omega^2 = m^2$.

The discussion of the causal structure properties of the Gödel–type spacetimes will be postponed into section 3.4.

3.2 The Lorentz force–free fluids

Because of cylindrical symmetry and stationarity there exist local coordinate systems $x^\mu = (t, \varphi,\ z,\ r)$ on the spacetime manifold, adapted to the Killing vector fields $\partial_t,\ \partial_\varphi,\ \partial_z$. Hypersurfaces $\varphi = 0$ and $\varphi = 2\pi$ are to be identified, r and z are the radial and longitudinal coordinates respectively. Since we demand the metric to be invariant over simultaneous time reversion $t \to -t$ and reflection $\varphi \to -\varphi$, and over the inversion $z \to -z$, the spacetime metric tensor[2] has the form

$$\mathrm{d}s^2 = \mathrm{e}^{2\alpha}\left(\mathrm{d}t + f\mathrm{d}\varphi\right)^2 - \mathrm{e}^{2\beta}\mathrm{d}\varphi^2 - \mathrm{e}^{2\gamma}\mathrm{d}z^2 - \mathrm{e}^{2\delta}\mathrm{d}r^2 \ , \tag{3.3}$$

where all functions $\alpha, \beta, \gamma, \delta$ and f depend only on the coordinate r.

Further calculations will be significantly simplified if one uses the orthonormal basis approach. For, it is convenient to introduce the tetrad basis 1–forms

$$\begin{aligned} \Theta^{\hat{0}} &= \mathrm{e}^{\alpha}(\mathrm{d}t + f\mathrm{d}\varphi) \ , \qquad & \Theta^{\hat{1}} &= \mathrm{e}^{\beta}\mathrm{d}\varphi \ , \\ \Theta^{\hat{2}} &= \mathrm{e}^{\gamma}\mathrm{d}z \ , & \Theta^{\hat{3}} &= \mathrm{e}^{\delta}\mathrm{d}r \ . \end{aligned} \tag{3.4}$$

All computations will be done in the basis (3.4) unless stated otherwise.

Let us turn to the physical content of the theory. Far the most often source for the cosmological models is the perfect fluid, and especially a dust. This is because at least according to present observational evidence the energy density dominates the pressure.

We look for the cosmological models so we can assume that our spacetime is filled with a perfect fluid that can be possibly charged as well.

The perfect fluid is uniquely characterised by the pressure p and the charge density μ, both this quantities depending on radial coordinate only, and by the velocity field u. Next progress usually consists of introducing the fluid comoving system in which the velocity vector field has simple form $u = \mathrm{e}^{-\alpha}\partial_t$.

[1] Of course there is the trivial case for $\Omega = m = 0$, which is just Minkowski spacetime with ten–parameter isometry group.

[2] In this thesis we the units are used in which $c = G = 1$.

It is well known that the velocity vector field gradient can be decomposed into several parts each of them describing the influence of the curvature on cloud of the fluid particles. The acceleration of the fluid is given by $\dot{u} = -\mathrm{d}\alpha$, so in general the fluid does not move along geodesic wordlines, unless α is constant. Because our problem is stationary, the expansion as well as shear tensor are vanishing identically. From physical point of view it means that a cloud of the particles will be temporarily neither expanding nor deformed. Very important quantity is the vorticity, which is essentially a measure of the cloud particles rotation. For our spacetime metric (3.3) the vorticity 1–form becomes

$$\omega = \frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}r}\mathrm{e}^{\alpha-\beta+\gamma-\delta}\mathrm{d}z \ .$$

In summary the fluid rotates as a rigid body around the z–axis, and generally its motion is not geodesical. Let us comment the electromagnetic field. At the time it will depend only on the radial coordinate. From the symmetry considerations, and by using the Einstein–Maxwell (EM) equations, the author was led to the following electromagnetic field 2–form. The only non–vanishing components are radial electric and longitudinal magnetic fields $\mathrm{E}_{\hat{r}}$ and $\mathrm{B}_{\hat{z}}$ respectively[3]. The case with a radially pointing magnetic field will be treated separately in subsection 3.2.1.

Very often, though not always, one postulates that the fluid particles are also charge carriers, which means that the current density j, that enters the Maxwell equations, is equal to $j = \rho u$, with ρ being the charge density.

Then from the Maxwell equations we obtain a coupling between the metric functions and electromagnetic field, called Maxwell condition, and one also gets an expression for the charge density.

Before constructing the EM equations it is useful to look at the Bianchi identity, that in our case reads

$$u \wedge * \left[\mu \mathrm{d}u + \mathrm{d}(p\ u) + \rho\ F\right] = 0 \ ,$$

or explicitly

$$\frac{\mathrm{d}p}{\mathrm{d}r} + (p+\mu)\frac{\mathrm{d}\alpha}{\mathrm{d}r} + \rho\frac{\mathrm{d}n}{\mathrm{d}r} = 0 \ . \tag{3.5}$$

The last term on the right–hand side (RHS) of (3.5) is proportional to the Lorentz force [8], and the remaining terms in fact arise from the Euler hydrodynamics equation.

The next step consists of constructing and solving the Einstein–Maxwell–fluid system. We find that we have an extra degrees of freedom, that correspond to the possibility of the radial rescaling, choosing the state equation $\mu = \mu(p)$ and fixing the charge density. One degree of freedom will be chosen in the following way: either α or γ is constant, this gives us two family of the solutions.

First let us consider the first class of the solutions, when α = const. We will omit here the most general solution [8] with non–vanishing Lorentz force, simply because of technical reasons. This solution is given in terms of integrals of certain functions. Instead

[3] Hatted indices denote the components in the orthonormal basis (3.4).

we have focused ourselves on the case when the Lorentz force vanishes, and the magnetic field survives only[4]. The explicit metric results in

$$\mathrm{d}s^2 = \left(\mathrm{d}t + \frac{2\Omega}{C}\gamma \mathrm{d}\varphi\right)^2 - \left(\frac{P}{C^2}\mathrm{e}^{2\gamma} + \frac{\lambda}{C}\gamma + \nu\right)\mathrm{d}\varphi^2 - \mathrm{e}^{2\gamma}\mathrm{d}z^2 - \frac{\gamma'^2 \mathrm{e}^{2\gamma}\mathrm{d}r^2}{P\mathrm{e}^{2\gamma} + C\lambda\gamma + C^2\nu}, \qquad \lambda = \frac{2B^2 - 2\Omega^2}{C}, \tag{3.6}$$

where we have four integration constants C, P, Ω, ν. The arbitrary function γ is non–constant and twice differentiable. As it shown in [10], the constant B is the magnitude of the longitudinal magnetic field on the rotation axis. Because of the Bianchi identity (3.5), the pressure is constant, while the energy density becomes

$$\mu = \frac{1}{4\pi}\left(2\Omega^2 - B^2\right)\mathrm{e}^{-2\gamma} - 3p\,. \tag{3.7}$$

This equation has very transparent physical interpretation. The magnetic energy density must be added to the "specific" energy density $\mu + 3p$ to balance the rotation.

Note that the fluid particles move along geodesic worldlines. Of course, since we are interested in cylindrically symmetric spacetimes, we have to impose the axial symmetry condition and the elementary flatness condition [7]. One usually also wants the energy conditions to be satisfied, which restricts values of the integration constants. Further discussion of the metric is found in [8] and in second part of the thesis.

Let us briefly mention the results for the second class, when γ is zero. Now, both the pressure as well as the energy density depend upon the radial coordinate. In this case the fluid particles motion is no longer geodesical, and this is because of the pressure inhomogeneity. It is seen immediately from the Bianchi identity. Again, the explicit formulae are given in [8].

3.2.1 A scalar field incorporation

For the future convenience we will discuss here the case when massless scalar field ϕ is considered in addition to the charged perfect fluid. The reason for this will become clear in section 4.3.

Let the Lorentz force acting on the fluid particles vanish. Then purely magnetic field is present. By excluding electric currents parallel to z–axis, one is led to the following electromagnetic field 2–form

$$F = \mathrm{B}_{\hat{z}}(r)\Theta^{\hat{r}} \wedge \Theta^{\hat{\varphi}} + \mathrm{B}_{\hat{r}}(r)\Theta^{\hat{\varphi}} \wedge \Theta^{\hat{z}}\,.$$

From the symmetry considerations it turns out that ϕ should depend on r as well as z coordinate. The radial magnetic field may seem rather artificial, but the form of the Einstein equations leads us to include it. Otherwise, from the Einstein equations, the

[4] For review of some Lorentz–force free (LFF) fluids in relativity see [9].

scalar field would be dependent either on r or z, but not simultaneously, $\frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial r} = 0$. After some rearrangements of the Einstein equations the scalar field becomes

$$\phi = \phi_0 + \phi_1 z + \phi_2 \int e^{-\beta-\gamma+\delta} dr \ ,$$

where the metric function e^{β} satisfies the following non-linear second-order equation

$$\frac{d}{dx}\left[\frac{\frac{du}{dx} + x}{u}\right] = \frac{4F_{\varphi z}^2 - 2\phi_2^2}{u^2} \ . \tag{3.8}$$

In (3.8) we have denoted $u \propto e^{2\beta}$, x is r–coordinate rescaled and $F_{\varphi z}$ is the φz–component of the electromagnetic field 2–form, expressed in the coordinate basis, which must be constant by virtue of the Maxwell equations $dF = 0$. Finally ϕ_2 is constant satisfying $\phi_1\phi_2 = 2BF_{\varphi z}$.

We proceed further by splitting the solutions of (3.8) into two groups according as the right–hand side of (3.8) vanishes or not. In Case I, when RHS does vanish, we obtain two different subcases [10]. First of them, when both the radially pointing magnetic field and ϕ_2 vanish, is a proper generalization of the solution (3.6) discussed above. On the other hand if the radial magnetic field does not vanish, one obtains already known solution found by Wright [11]. But now we have an alternative source for this metric, since Wright considered only perfect fluid as the matter content of his spacetime. In our case, however, the combination of massless scalar field with the electromagnetic one results in the spacetime, which looks precisely like Wright solution. What is not surprising is that now one has magnetic charges (or monopoles), situated on the rotation axis. Fortunately they also carry a scalar charge which cancels the magnetic one, so that the stress–energy tensor is regular on the rotation axis.

We mention only briefly the Case II, when the left–hand side of (3.8) does not vanish, so the radially pointing magnetic field is not compensated by the scalar field gradient radial part. This solution can not be regular on the rotation axis, and gives us the van Stockum metric alternative source. Much more detailed discussion of the obtained results is found in [10].

3.3 Components of the Riemann tensor and field equations

3.3.1 Riemann tensor components

Probably the most efficient method for computing the connection and curvature for our general basis (3.4) is to use the Cartan structure equations. Let TM be the tangent bundle over M. Choose local coframe fields $\Theta^{\hat{\alpha}}$, that constitute local cross–sections of the cotangent bundle T^*M. Let $\omega^{\hat{\alpha}}{}_{\hat{\beta}}$ and $\Omega^{\hat{\alpha}}{}_{\hat{\beta}}$ are connection 1–forms and curvature 2–forms respectively, i.e. the Lie algebra $so(3,1)$ valued differential forms on M. They satisfy the Cartan equations

$$\mathrm{d}\Theta^{\hat{\alpha}} + \omega^{\hat{\alpha}}{}_{\hat{\beta}} \wedge \Theta^{\hat{\beta}} = 0 \tag{3.9a}$$

$$\Omega^{\hat{\alpha}}{}_{\hat{\beta}} = \mathrm{d}\omega^{\hat{\alpha}}{}_{\hat{\beta}} + \omega^{\hat{\alpha}}{}_{\hat{\gamma}} \wedge \omega^{\hat{\gamma}}{}_{\hat{\beta}} = 0\ , \tag{3.9b}$$

subject to the equations

$$\mathrm{d}g_{\alpha\beta} = \omega_{\hat{\alpha}\hat{\beta}} + \omega_{\hat{\beta}\hat{\alpha}} \tag{3.10}$$

and

$$\Omega^{\hat{\alpha}}{}_{\hat{\beta}} = \frac{1}{2} R^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}\hat{\delta}} \Theta^{\hat{\gamma}} \wedge \Theta^{\hat{\delta}}\ . \tag{3.11}$$

If we denote for the future convenience

$$\begin{array}{ll} g = f'' \mathrm{e}^{\alpha-\beta-2\delta}\ , & A = (\alpha'' + \alpha'^2 - \alpha'\delta')\,\mathrm{e}^{-2\delta}\ , \\ f = f' \mathrm{e}^{\alpha-\beta-\delta}\ , & B = (\beta'' + \beta'^2 - \beta'\delta')\,\mathrm{e}^{-2\delta}\ , \\ a = \alpha' \mathrm{e}^{-\delta}\ , & C = (\gamma'' + \gamma'^2 - \gamma'\delta')\,\mathrm{e}^{-2\delta}\ , \\ b = \beta' \mathrm{e}^{-\delta}\ , & F = \frac{1}{4} f'^2 \mathrm{e}^{2\alpha-2\beta-2\delta}\ , \\ c = \gamma' \mathrm{e}^{-\delta}\ , & \end{array} \tag{3.12}$$

the connection 1–forms can be determined from (3.9a). For (3.3) they are equal to

$$\begin{array}{ll} \omega^{\hat{t}}{}_{\hat{\varphi}} = -\frac{1}{2} f \Theta^{\hat{r}}\ , & \omega^{\hat{t}}{}_{\hat{r}} = \frac{1}{2} f \Theta^{\hat{\varphi}} + a\ \Theta^{\hat{t}}\ , \\ \omega^{\hat{\varphi}}{}_{\hat{r}} = -\frac{1}{2} f \Theta^{\hat{t}} + b\ \Theta^{\hat{\varphi}}, & \omega^{\hat{z}}{}_{\hat{r}} = c\ \Theta^{\hat{z}}\ . \end{array} \tag{3.13}$$

All other components are either vanishing or related to these by (3.10). Similarly Cartan equations (3.9b) and (3.11) yield the following non–vanishing components of the Riemann tensor

$$\begin{array}{ll} R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -(A+F)\ , & R_{\hat{t}\hat{z}\hat{\varphi}\hat{z}} = -\frac{1}{2} fc\ , \\ R_{\hat{t}\hat{\varphi}\hat{t}\hat{\varphi}} = -(ab+F)\ , & R_{\hat{t}\hat{z}\hat{t}\hat{z}} = -ac\ , \\ R_{\hat{\varphi}\hat{r}\hat{\varphi}\hat{r}} = B - 3F\ , & R_{\hat{\varphi}\hat{z}\hat{\varphi}\hat{z}} = bc\ , \\ R_{\hat{t}\hat{r}\hat{\varphi}\hat{r}} = -\frac{1}{2}[g + f(3a - b - d)]\ , & R_{\hat{r}\hat{z}\hat{r}\hat{z}} = C\ . \end{array} \tag{3.14}$$

For the components of the Ricci tensor we have

$$\begin{array}{l} R_{\hat{t}\hat{\varphi}} = \frac{1}{2}[g + f(3a - b + c - d)]\ , \\ R_{\hat{t}\hat{t}} = A + ab + ac + 2F\ , \\ R_{\hat{\varphi}\hat{\varphi}} = -B + ab + bc + 2F\ , \\ R_{\hat{z}\hat{z}} = -C + ac + bc\ , \\ R_{\hat{r}\hat{r}} = -A - B - C + 2F\ . \end{array} \tag{3.15}$$

Finally the scalar curvature is given by

$$R = 2(A + B + C - F + ab + bc + ca)\ . \tag{3.16}$$

3.3.2 Variational approach to the equations of motion

In this subsection we briefly study the variational approach in our case, when a charged perfect fluid is coupled to gravity. The action S is proportional to

$$S \sim \int (*R + 16\pi * \mu + 2F \wedge *F) \ , \tag{3.17}$$

provided that the electromagnetic source term can be transformed away.

There is an economical method of constructing the Einstein equations. It uses the Cartan structure equations and was discussed in subsection 3.3.1. Once we have the scalar curvature for the metric (3.3), we may quickly proceed as follows. According to (3.16), the scalar curvature becomes

$$\begin{aligned} R = 2\Big(&\alpha'' + \beta'' + \gamma'' + \alpha'^2 + \beta'^2 + \gamma'^2 + \alpha'\beta' + \beta'\gamma' + \gamma'\alpha' \\ &-\alpha'\delta' - \beta'\delta' - \gamma'\delta'\Big)\, \mathrm{e}^{-2\delta} - \frac{1}{2} f'^2 \mathrm{e}^{2(\alpha+\beta-\delta)} \ . \end{aligned} \tag{3.18}$$

Because the Einstein-Hilbert Lagrangian $R\sqrt{-g}$ can be written in the form

$$\begin{aligned} R\sqrt{-g} = &-2\left(\alpha'\beta' + \beta'\gamma' + \gamma'\alpha'\right) \mathrm{e}^{\alpha+\beta+\gamma-\delta} - \frac{1}{2} f'^2 \mathrm{e}^{(3\alpha-\beta+\gamma-\delta)} \\ &+2\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\alpha' + \beta' + \gamma'\right) \mathrm{e}^{\alpha+\beta+\gamma-\delta}\right] \ , \end{aligned} \tag{3.19}$$

and the total derivative does not contribute to the action, one can drop the last term on the right–hand side. Now let us consider the variation of the energy density μ. The matter current vector[5] j is expressed in terms of the velocity and mass density as $j = \rho u$. We have

$$\rho^2 = -\frac{1}{g}\left(\sqrt{-g} j^\alpha \sqrt{-g} j^\beta\right) g_{\alpha\beta} \ ,$$

which for our metric in the fluid comoving system reduces to

$$\rho^2 = -\mathrm{e}^{-2(\beta+\gamma+\delta)} \left[\mathrm{e}^{2(\beta+\gamma+\delta)} (u^0)^2\right] \ , \tag{3.20}$$

where the expression in the square brackets remains invariant with respect to the variation. From (3.20) one obtains

$$\delta\rho = -\rho\,(\delta\beta + \delta\gamma + \delta\delta) \ . \tag{3.21}$$

The energy density is related to the mass density ρ and the (relative) internal energy ϵ by

$$\mu = \rho\,(1 + \epsilon) \ ,$$

from which one gets

$$\delta\mu = \frac{\delta\rho}{\rho}\,(\mu + p) \ . \tag{3.22}$$

[5] This is the only section in which the fluid flow vector is denoted by the same letter as the electric current vector and the mass density as the charge density.

Inserting (3.21) into the (3.22) gives following desired result

$$\delta L_{\text{fluid}} = \delta\left(2\mu\sqrt{-g}\right) = 2\mu\delta\alpha - 2p\left(\delta\beta + \delta\gamma + \delta\delta\right)\mathrm{e}^{\alpha+\beta+\gamma+\delta}\ . \tag{3.23}$$

With the explicit expressions (3.19) and (3.23), the variation of the action (3.17) is straightforward, and the complete Einstein plus matter equations are written down in papers [8, 10] of second part.

3.4 Chronology violation

In this work we will restrict ourselves to entirely classical examination of the chronology violation. The questions concerning the time travel thermodynamics, statistics and quantum mechanics will not be discussed in this thesis. In general it can be said that the chronology is violated if the light cones are sufficiently tipped over in a spacetime region. Conversely, if the light cones point out in nearly the same direction, the chronology will be protected. This light cones tipping over is the common property of the causality violating spacetimes.

It should be mentioned, that in all cases described in this section, the chronology violation is not a consequence of the spacetime multiconnectness, as it was the case in chapter 2. It was shown that it was easy to construct spacetimes containing CTCs, if the multiconnectness was permitted (for example the cylindrical universe). However, it was always possible to take universal covering manifold which is simply connected by definition in those examples. It means that CTCs that were formed out due to the spacetime multiconnectness will be ruled out. Such CTCs which can be ruled out by taking the universal covering spacetime are called trivial (in contrast with the non–trivial CTCs) and we will not concern them furthermore. Our solutions that were found in previous section are simply connected.

We have seen that if the vorticity vector of the fluid does not vanish, there is no family of spacelike hypersurfaces everywhere orthogonal to the fluid particles worldlines. The case with the vanishing vorticity corresponds to a static spacetime, and there are no CTCs. If the vorticity does not vanish, it is not possible to patch together 3–spaces orthogonal to the velocity field at every spacetime point, in order to form a family of hypersurfaces. Thus the necessary (but not a sufficient) condition for the existence of the CTCs is fulfilled. For, if it was possible to patch the 3–spaces smoothly into integral submanifolds, there would exist the time function, which would be constant on them (and not increasing).

The presence of the CTCs in a spacetime whose metric is written down in the cylindrical coordinates may be indicated by the norm of the vectors tangent to the circles $\{t, z, r\} =$ const, i.e. to the integral trajectories of the vector field ∂_φ.

From chapter 2 we know that if a given spacetime contains one CTC, then it contains an infinite number of CTCs. Another property is that if a spacetime contains a CTC, it also contains closed null geodesic.

Nevertheless, the presence of CTCs does not mean that the spacetime is not time orientable. As the consequence, anyhow one travels, ones time direction agrees with the local arrow of time at any point. Hence, we may define timelike or null vector w as future directed if its scalar product with the vector ∂_t is positive.

For concreteness and demonstration we examine the chronological structure of the metric (3.6) for the following specialization of the integration constants

$$2P = -C^2\nu = m^2 > 0,\ \gamma = 2Cm^{-2}\mathrm{sh}^2\left(\frac{mr}{2}\right) \equiv \tilde{C}\mathrm{sh}^2\left(\frac{mr}{2}\right),$$

where m is of inverse length dimension constant. The metric (3.6) becomes

$$\begin{aligned} \mathrm{d}s^2 &= \left[\mathrm{d}t + \tfrac{4\Omega}{m^2}\mathrm{sh}^2\left(\tfrac{mr}{2}\right)\mathrm{d}\varphi\right]^2 - \mathrm{e}^{2\tilde{C}\mathrm{sh}^2\left(\frac{mr}{2}\right)}\mathrm{d}z^2 \\ &\quad -\tfrac{2}{\tilde{C}^2m^2}\left[\mathrm{e}^{2\tilde{C}\mathrm{sh}^2\left(\frac{mr}{2}\right)} - 2\tilde{C}(1-\tilde{C})\mathrm{sh}^2\left(\tfrac{mr}{2}\right) - 1\right]\mathrm{d}\varphi^2 \\ &\quad -\frac{\tilde{C}^2}{2}\frac{\mathrm{sh}^2(mr)\mathrm{e}^{2\tilde{C}\mathrm{sh}^2\left(\frac{mr}{2}\right)}\mathrm{d}r^2}{\mathrm{e}^{2\tilde{C}\mathrm{sh}^2\left(\frac{mr}{2}\right)} - 2\tilde{C}(1-\tilde{C})\mathrm{sh}^2\left(\frac{mr}{2}\right) - 1}, \end{aligned} \tag{3.24}$$

where $2\Omega^2 = 2B^2 + m^2\left(1-\tilde{C}\right)$ and $\tilde{C} \leq 0$ by virtue of energy conditions [8]. Note that in $C \to 0$ regime, the equation (3.24) gives the Gödel–type metric (3.1), corresponding to letter b).

Clearly the projection of the velocity u into the z and r–directions vanishes, since $g_{tz} = g_{tr} = 0$. The only non–vanishing cross–term of the metric tensor, $g_{t\varphi}$, is increasing function of the radial coordinate. At the origin $r = 0$, $g_{t\varphi}$ vanishes and the velocity points out in the time direction. With increasing distance from the rotation axis, the projection of u into the φ–direction becomes non–zero, so the fluid particles trajectories are helicoids. The further from the rotation axis, the greater is the projection of u into φ–direction. Equivalently, the further from the rotation axis, the more significant is the light cones tipping over into the rotation direction, i.e. direction of increasing φ. For certain R, determined as a positive root of $g_{\varphi\varphi}$, the vector ∂_φ is null and the light cones are tangent to the plane (φ, r). Moving apart from the rotation axis results in the vector ∂_φ becoming timelike future directed, since $g_{t\varphi} > 0$. The light cones intersect the (φ, r) plane, see figure 3.1. The analysis shows that the magnetic field has positive influence on the chronology violation. If the circles $\{t, z, r\} = \mathrm{const}, r > R$ are CTCs, $\mathrm{d}s^2 = g_{\varphi\varphi}\mathrm{d}\varphi^2 > 0$, then it follows that even the curves

$$\{t = -k\varphi, z, r = \mathrm{const}, r > R, \varphi \in [0, 2\pi)\} \tag{3.25}$$

are timelike, because $\mathrm{d}s^2 = (k^2 g_{tt} - 2kg_{t\varphi} + g_{\varphi\varphi})\,\mathrm{d}\varphi^2 > 0$ for sufficiently small k. The vector, w tangent to the curves (3.25), $w = -k\partial_t + \partial_\varphi$, is for sufficiently small positive k future directed, since $u \cdot w = -kg_{tt} + g_{t\varphi} > 0$. Thus the end point of the curve at $\varphi = 2\pi$ chronologically precedes the starting point at $\varphi = 0$.

In this way it is possible to reach an arbitrary distant past event by spiralling downwards against the coordinate time (but in general settings, it is not possible to travel into a time before the origin of the time machine).

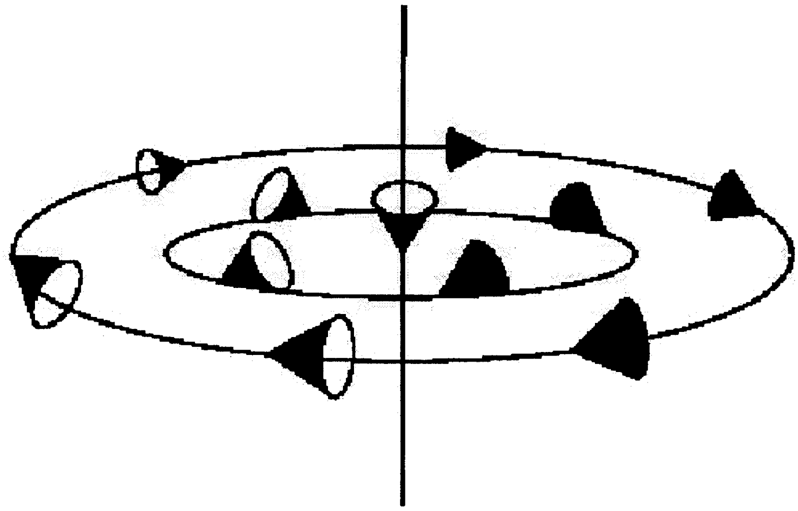

Figure 3.1: Light cones tipping over

The solution found does not contain any horizons (removable singularities), in particular no chronological horizons. Therefore if for any point with the radial coordinate r_1 the inequality $g_{\varphi\varphi}(r_1) > 0$ holds, the chronology violating region I^0 coincides with the entire spacetime. For, consider an arbitrary point p. Because of the horizons absence one can join p with a point q lying on a timelike circle $\{t, z = \text{const}, r = r_1\}$ by a future directed timelike curve γ. Then γ can be joined with the future directed timelike curve λ parametrically given by $\{t = -k\varphi, \varphi, z = \text{const}, r = r_1\}$. Along λ one can reach a point q' such that $p \in I^+(q')$. But because simultaneously $p \ll q$ and $q \ll q'$, we conclude that $q' \in I^+(p)$ and $p \in I^+(p)$. Because of the absence of any horizons, this construction applies to any spacetime point.

So far we have especially considered the CTCs, that are circles centred on the rotation axis. But one can naturally ask whether a spacetime may violate the chronological condition even though the circles above are not CTCs. The answer is no. More precisely, there are no CTCs present if $g_{\varphi\varphi} \leq 0$ everywhere. A simple proof was first discovered by Maitra [12]. For, suppose there is at least one CTC, say γ. Since γ is closed, the time coordinate along it is periodic function of the proper time. Let τ_0 be the value of the proper time τ, for which $t(\tau_0)$ attains the maximum (or minimum). Thus $\frac{\mathrm{d}t}{\mathrm{d}\tau}\Big|_{\tau_0} = 0$ and

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2_{\tau_0} = g_{\varphi\varphi}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2_{\tau_0} + g_{zz}\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2_{\tau_0} + g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2_{\tau_0} \leq 0 \ ,$$

since both g_{zz} and g_{rr} are negative because of the correctness of the Lorentz signature, and $g_{\varphi\varphi}$ is non–positive by the assumption. So the tangent vector is not timelike at $\tau = \tau_0$, and γ can not be everywhere timelike curve.

The explicit and practical criterion for existence or non–existence of CTCs gives the Carter's theorem [13, 14], which applies to any connected spacetime with the abelian timewise orthogonally transitive isometry group. The theorem states that if there is no Lie algebra covector such that the corresponding differential form is well behaved and timelike on each surface of transitivity, than the whole spacetime coincides with its chronology violating region. In other words, any two points can be joined each other by both future and past directed timelike curve. It can be easily proved that the Carter's statement is equivalent to the previous one, if the spacetime is cylindrically symmetric, and its metric is written down in the cylindrical coordinates. In our case the isometry group generated by the Killing vectors ∂_t, ∂_φ and ∂_z is timewise orthogonally transitive through the entire spacetime, the surfaces of transitivity being the timelike hypersurfaces $r = \text{const}$. Choosing the differential form corresponding to the Lie algebra covector means to choosing the differential form

$$\psi = a \ \mathrm{d}t + b \ \mathrm{d}\varphi + c \ \mathrm{d}z \ , \tag{3.26}$$

with the constants a, b and c. A simple algebra reveals that metric (3.3) contains CTCs if and only if $g_{\varphi\varphi} > 0$.

Now, let us return to the Gödel–type spacetimes (3.1) and consider for example the case b) when

$$g_{\varphi\varphi} = \frac{4}{m^2}\mathrm{sh}^2\left(\frac{mr}{2}\right)\left[\left(\frac{4\Omega^2}{m^2} - 1\right)\mathrm{sh}^2\left(\frac{mr}{2}\right) - 1\right] \ . \tag{3.27}$$

From (3.27) it is seen immediately that there are no CTCs if $4\Omega^2 \leq m^2$. Moreover, it can be shown [3] that the metric is stably causal for $4\Omega^2 < m^2$. In contrast to case b), the cases a) and c) always violate the chronology condition.

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Chapter 4

Impulses from quantum gravity

4.1 Gauss–Bonnet theorem

The Gauss-Bonnet theorem is an important tool in differential topology since it gives a connection between purely topological and geometrical (or analytical) quantities. Originally it has been formulated for orientable two-dimensional manifolds – Riemann surfaces, but it has been generalized to an arbitrary even–dimensional manifolds. The topological quantity appearing in the Gauss–Bonnet theorem is the Euler characteristic χ. It is not subject of this work to study it in details, so we only mention the known formula [1], coming from differential topology, how the Euler characteristics of a compact m–dimensional manifold M can be computed in terms of the manifold cohomology classes $\mathbb{H}^k(M,\mathbb{Z})$,

$$\chi(M) = \sum_{0\leq k\leq m} (-1)^k \mathrm{dim}\mathbb{H}^k(M,\mathbb{R}) \ .$$

It can be easily shown that χ vanishes identically if m is odd.

Having defined the Euler characteristics we may write down the Gauss–Bonnet theorem. Let M be a compact $2n$-dimensional oriented manifold. Choose local coframe fields (i.e. an orthonormal basis of 1-forms, indices are not hatted) $\Theta^i, i = 1,..,2n$ whose curvature 2-forms are Ω_{ij}. Then

$$\chi(M) = \int_M e(M) \ , \tag{4.1}$$

where the Euler class $e(M)$ of the manifold M is defined[1] in terms of the curvature 2-forms on the tangent bundle TM by

$$e(M) = \frac{(-1)^n}{2^{2n}\pi^n n!}\epsilon_{i_1 i_2 \dots i_{2n-1} i_{2n}} \Omega_{i_1 i_2} \wedge \dots \wedge \Omega_{i_{2n-1} i_{2n}} \ . \tag{4.2}$$

Here $\epsilon_{i_1..i_{2n}}$ are components of the volume element in the orthonormal basis Θ^i. The Gauss–Bonnet theorem (4.1) can be derived either directly or from a so called index theorem [1].

[1]This is the convention usually found in the literature. To be more precise, the Euler class is defined for the tangent bundle TM over M rather then for M itself.

For the physically interesting situation in which $n = 2$, i.e. four–dimensional spacetime, the equations (4.1) and (4.2) take the form

$$\chi = \frac{1}{32\pi^2} \int \left(R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2 \right) \eta \; , \tag{4.3}$$

where η is the volume element.

A comprehensive introduction to differential topology, characteristic classes and the index theorem can be found for example in [1, 2, 3].

4.2 Renormalization and perturbation theory

This section is intended to clarify the reasons leading to the inclusion of higher–derivative terms into the ordinary Einstein–Hilbert action. We closely follow the review [4].

The basic idea of the perturbation theory is to split a metric field into two pieces. One of them, $g^{(0)}$, is viewed as a classical background field, and the second one, h is a quantum correction, or in other words, a quantum field that propagates on the background $g^{(0)}$. One can expand the action $I[g]$ in powers of h like

$$I[g^{(0)} + h] = I[g^{(0)}] + \left.\frac{\delta I}{\delta g^{\alpha\beta}}\right|_{g=g^{(0)}} \delta h_{\alpha\beta} + \frac{1}{2} \left.\frac{\delta^2 I}{\delta g_{\alpha\beta} \delta g_{\mu\nu}}\right|_{g=g^{(0)}} \delta h_{\alpha\beta} \delta h_{\mu\nu} + \dots \; . \tag{4.4}$$

The first term on the right–hand side is the classical action for background $g^{(0)}$. The second term vanishes by virtue of Einstein equations[2]. The third term represents the gravitational field propagator, and the dots stand for the higher–order terms responsible for the gravitational interaction.

A simple dimensional analysis shows that all terms in the expansion (4.4) contain two derivatives. Therefore the typical propagator goes as k^{-2}, where k is the momentum of the graviton.

We outline here the reason for the incorporation of terms quadratic in curvature. Consider a Feynmann diagram with E external lines, I internal lines, L loops and V_n n–point vertices. Altogether the internal lines contribute to the amplitude by an amount k^{-2I}. Every loop is to be integrated over, so the total loop contribution is proportional to $\mathrm{d}^{4L}k$. Finally, each vertex contributes by the factor k^2. *Primitive degree of divergence* D of the diagram is defined as

$$D = -2I + 4L + 2 \sum_{n \geq 3} V_n \; . \tag{4.5}$$

If a momentum space cut–off Λ is imposed, the diagram diverges like Λ^D as Λ tends to infinity. If D is unbounded above, the theory is generally non–renormalizable.

Two following topological identities are useful in decision whether a given diagram is renormalizable or not. First, since each internal line has two ends and each external line

[2] In this section we consider the vacuum case. Matter fields could be added straightforwardly.

has just a single one, one immediately obtains the relation

$$2I + E = \sum_n nV_n \ . \tag{4.6}$$

The second identity appears when one counts the number of loops in a diagram. It equals the number of undetermined momenta which have to be integrated over. Thus it holds[3]

$$L = I - E - \sum_n V_n \ . \tag{4.7}$$

Inserting (4.7) into (4.5) yields

$$D = 2L - 2E \ ,$$

and so the primitive degree of divergence increases unboundedly with number of loops.

The standard regularization method – dimensional regularization – gives rise to the following one–loop corrected counter–term [4]

$$\frac{1}{\epsilon}\frac{1}{2880\pi^2}\int R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\eta \ , \tag{4.8}$$

where $\epsilon = 4 - n$, with n being the spacetime dimension. Assuming that the Einstein equations $R_{\alpha\beta} = 0$ are satisfied, one can in fact drop (4.8) from the action, since from the Gauss–Bonnet theorem (4.3) one has

$$\begin{aligned} \chi = & \ \tfrac{1}{32\pi^2}\int \left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2\right)\eta + \text{b.t.} \\ = & \ \tfrac{1}{32\pi^2}\int R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\eta + \text{b.t.} \ , \end{aligned}$$

where b.t. stands for boundary terms[4]. However, the Euler characteristic χ is topological invariant and does not contribute to the dynamics. Thus one–loop corrected general relativity (so called semi–classical gravity) is well–behaved.

Nevertheless the hope that the theory remains well–defined at two–loop level is lost since the corresponding counter–term takes the form [5]

$$\frac{1}{\epsilon}\frac{209}{737280\pi^4}\int R_{\alpha\beta}{}^{\gamma\delta}R_{\gamma\delta}{}^{\lambda\tau}R_{\lambda\tau}{}^{\alpha\beta}\eta \ ,$$

and there is no analogy with the Gauss–Bonnet theorem.

As it is shown in [4], one possible way to modify the classical gravitational action to be renormalizable is adding the terms quadratic in the curvature. That this incorporation leads to the renormalizable theory is indicated as follows [4]. These theories contain fourth derivatives of the metric and according to the discussion above, the propagator goes as k^{-4}. The primitive degree of divergence becomes

$$D = -4I + 4L + 4\sum_n V_n \ .$$

If the use of (4.7) is made, one concludes that $D = -4E$, and so the theory is renormalizable. But it suffers from another disaster – contains a tachyon in its spectrum [4].

[3] See comment in [4].

[4] In general we consider here the manifold with boundary.

4.2.1 Terms quadratic in the Riemann tensor and their variations

In this subsection, within the variational approach, we give the results for the variations of three basic and most often used higher-order derivative terms. These are $R^2, R_{\alpha\beta}R^{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$. We derive the corresponding expression only for the latter term, since the derivation of the remaining ones is even simpler.

By using the Palatini equation

$$\delta R^{\alpha}{}_{\beta\gamma\delta} = \left(\delta\Gamma^{\alpha}_{\beta\delta}\right)_{;\gamma} - \left(\delta\Gamma^{\alpha}_{\beta\gamma}\right)_{;\delta}$$

one straightforwardly obtains

$$\frac{\delta}{\delta g^{\mu\nu}}\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\right) = 2R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma} + 4R_{\mu\alpha\nu\beta}{}^{;\beta\alpha} . \tag{4.9}$$

It can be seen by using the Ricci identity that the second term on the right-hand side of equation (4.9) is symmetric in indices μ and ν. The Bianchi identity applied on this term gives

$$\begin{aligned} R_{\mu\alpha\nu\beta}{}^{;\beta\alpha} &= R_{\mu\nu}{}^{;\alpha}{}_{;\alpha} - R_{\nu\alpha;\mu}{}^{;\alpha} \\ &= R_{\mu\nu}{}^{;\alpha}{}_{;\alpha} - R_{\nu\alpha}{}^{;\alpha}{}_{;\mu} + R_{\mu\alpha\nu\beta}R^{\alpha\beta} - R_{\mu\alpha}R_{\nu}{}^{\alpha} \\ &= R_{\mu\nu}{}^{;\alpha}{}_{;\alpha} - \frac{1}{2}R_{;\mu\nu} + R_{\mu\alpha\nu\beta}R^{\alpha\beta} - R_{\mu\alpha}R_{\nu}{}^{\alpha} . \end{aligned} \tag{4.10}$$

By performing a calculation similar to those leading to (4.10) repeated for $R_{\alpha\beta}R^{\alpha\beta}$ and R^2, and taking the pseudoscalar $\sqrt{-g}$ into account, one arrives at the following expressions

$$\frac{\delta}{\delta g^{\mu\nu}}\left(R^2\sqrt{-g}\right) = 2RR_{\mu\nu} - 2R_{;\mu\nu} + 2g_{\mu\nu}R_{;\rho}{}^{;\rho} - \frac{1}{2}R^2 g_{\mu\nu} , \tag{4.11}$$

$$\begin{aligned} \frac{\delta}{\delta g^{\mu\nu}}\left(R_{\alpha\beta}R^{\alpha\beta}\sqrt{-g}\right) &= 2R_{\mu\rho\nu\tau}R^{\rho\tau} - R_{;\mu\nu} + R_{\mu\nu;\rho}{}^{;\rho} + \frac{1}{2}g_{\mu\nu}R_{;\rho}{}^{;\rho} \\ &\quad -\frac{1}{2}R_{\rho\tau}R^{\rho\tau}g_{\mu\nu} , \end{aligned} \tag{4.12}$$

$$\begin{aligned} \frac{\delta}{\delta g^{\mu\nu}}\left(R_{\alpha\beta\rho\tau}R^{\alpha\beta\rho\tau}\sqrt{-g}\right) &= 2R_{\mu\rho\lambda\tau}R_{\nu}{}^{\rho\lambda\tau} + 4R_{\mu\rho\nu\tau}R^{\rho\tau} - 4R_{\mu\rho}R_{\nu}{}^{\rho} \\ &\quad -2R_{;\mu\nu} + 4R_{\mu\nu;\rho}{}^{;\rho} - \frac{1}{2}R_{\alpha\beta\rho\tau}R^{\alpha\beta\rho\tau}g_{\mu\nu} . \end{aligned} \tag{4.13}$$

Putting (4.13), (4.12)and (4.11) together, we get the formula for variation of the Gauss–Bonnet term

$$R^2_{GB} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2 \tag{4.14}$$

appearing in the four-dimensional Gauss–Bonnet theorem (4.3), which follows from

$$\begin{aligned} \frac{\delta}{\delta g^{\mu\nu}}\left(R^2_{GB}\sqrt{-g}\right) &= 2\left(R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma} - 2R_{\mu\alpha\nu\beta}R^{\alpha\beta}\right. \\ &\quad \left. -2R_{\mu\alpha}R_{\nu}{}^{\alpha} + RR_{\mu\nu}\right) - \frac{1}{2}g_{\mu\nu}R^2_{GB} . \end{aligned} \tag{4.15}$$

On the other hand, because the Euler class (4.2) defined in four–dimensions by the Gauss–Bonnet term (4.14) is a topological invariant, it can not be affected by a small metric variation. Hence

$$\frac{\delta}{\delta g^{\mu\nu}}\,(R^2_{GB}\sqrt{-g}) = 0\ ,$$

and one gets an interesting identity by setting the right–hand side of (4.15) equal to zero.

4.3 Lorentz force–free fluids in string cosmology

Except of the study of the exact solutions to the Einstein field equations, the author has attempted to treat a problem of existence of cylindrically symmetric spacetimes (3.3) within the framework of string cosmology (see [13] and the references therein).

In the last decade string cosmology has become an attractive subject of interest. As one goes to low energies, string cosmology is actually classical cosmology of general relativity enriched by addition of massless scalar fields and possibly 1–form gauge potentials giving rise to 2–form field strengths.

Historically, scalar fields were incorporated into the theory for various reasons. Starting at Brans–Dicke cosmology [6] the progress continuated up to systematic study of general relativistic scalar fields. Also in the framework of Gödel–type solutions there was an effort to examine the properties and behaviour of scalar fields in combination with other sources. Chakraborty and Bandyopadhyay [7] examined general scalar field for Gödel–type spacetime. Rebouças and Tiomno considered some of the possible sources for Gödel–type solution with five and seven–parameter isometry group with the help of the Sergè classification of the Ricci tensor [8, 9]. In a series of papers [10]–[12] Accioly has given interesting Gödel–type solutions with and without closed timelike curves in case of higher–derivative gravity. Finally Barrow [13] and then Kanti and Vayonakis have found Gödel–type solutions following from the string effective action. In all but one of the above cases, by virtue of spacetime homogeneity in local cylindrical coordinates $(t,\ \varphi, z,\ r)$ the scalar fields depend linearly on the z–coordinate. In fact only the axion in [14] has rich dependence on r as well as z coordinates.

In string theory there appears new fundamental constant α', whose physical meaning is that of the inverse string tension [15]. It plays a role analogical to the role of Planck constant in the ordinary quantum mechanics. In particular, α' serves as a parameter with respect to which the worldsheet–loop expansions are to be carried out [15].

Within frame of 1–loop corrected string–inspired generalized theory of gravity the effective action can be expressed as the following sum[14]

$$S_{eff} = S_{EH} + S_{corr} + S_{sc} + S_{elmag}\ . \tag{4.16}$$

The first term in (4.16) is the ordinary Einstein–Hilbert action[5] $S_{EH} = \int *R$, where R is the Ricci curvature of the spacetime metric g. A contribution S_{sc} of the scalar fields to the

[5]For the convenience we have omitted the constant factor $(16\pi)^{-1}$ standing in front of the action integrals in this section.

effective action (4.16) is given by

$$S_{sc} = -\sum_i \int \left(\mathrm{d}\phi_i \wedge *\mathrm{d}\phi_i + \mathrm{e}^{-2\phi_i} \mathrm{d}a_i \wedge *\mathrm{d}a_i \right) ,$$

where ϕ_i and a_i are scalar fields, i runs from 1 to some integer N. Physically $\phi_1 \equiv \phi$ and $a_1 \equiv a$ represent the dilaton and the axion respectively, while ϕ_i and a_i, $i \neq 1$, are modulus fields. Stringy correction S_{corr} to the Einstein–Hilbert action is

$$S_{corr} = \varepsilon \int \left(\mathrm{e}^{\phi} * R^2_{GB} + 4a \ \mathrm{Tr}\ \Omega \wedge \Omega \right) , \tag{4.17}$$

where $\varepsilon = \frac{\alpha'}{4\mathrm{g}^2}$ with g being the Yang–Mills constant[6]. The second term in (4.17) can be rewritten in terms of the Riemann tensor as

$$\mathrm{Tr}\ \Omega \wedge \Omega = -\frac{1}{4} R^{\alpha\beta}{}_{\gamma\delta} R^{\sigma\tau}{}_{\alpha\beta}\ \eta^{\gamma\delta}{}_{\sigma\tau}\ \eta , \tag{4.18}$$

with the volume element $\eta = \sqrt{-g}\ \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$.

If we turn to our case (3.3), a straightforward, although tedious, calculation gives the expression for the Gauss–Bonnet term

$$R^2_{GB} = 2\mathrm{e}^{-\alpha-\beta-\gamma-\delta} \frac{\mathrm{d}}{\mathrm{d}r} \left[\left(f'^2 \gamma' \mathrm{e}^{2(\alpha-\beta)} + 4\alpha'\beta'\gamma' \right) \mathrm{e}^{\alpha+\beta+\gamma-3\delta} \right] , \tag{4.19}$$

together with $R^{\kappa\lambda}{}_{\mu\nu} * R_{\kappa\lambda}{}^{\mu\nu} = 0$ implying that the term (4.18) does not contribute to (4.16).

Finally we write down the contribution S_{elmag} of the electromagnetic field

$$S_{elmag} = \varepsilon \int \left(2\mathrm{e}^{\phi} F \wedge *F - 4a\ F \wedge F \right) . \tag{4.20}$$

We are interested in stationary cylindrically symmetric spacetimes whose metric can be written in the form (3.3). What is missing in (4.16), and what one would certainly want, are fermions. So our *Ansatz* is that at least on this semi–classical level, the fermions can be thought as a perfect fluid with the energy density μ and the pressure p. Thus the perfect (fermionic) fluid is a rough approximation of the presence of the fermions in the theory. It means that in (4.16) one has an additional term $S_{fluid} = \int 16\pi * \mu$ associated with the perfect fluid.

Now the question of the existence of the charged perfect fluid coupled to the electromagnetic fields is addressed.

4.3.1 Einstein–Maxwell–dilaton gravity

The field equations for metric (3.3) following from the action (4.16) are rather complicated, especially because of the presence of the Gauss–Bonnet term (4.19). Therefore the simplest

[6]To be more precise, the physical meaning of g depends on details of compactification [15].

way is to find a solution within a framework of Einstein–Maxwell–dilaton gravity, when the Gauss–Bonnet term is omitted.

This is accomplished in works [16, 17], where the method of iteractions with respect to the parameter ε was used. Thus, at zero–order, the solution found describes the purely classical uncharged dilaton plus fluid gravitating system. Then it is generalized to ε–order by taking the action (4.16) up to the terms linear in ε.

For details, the reader is referred in particular to work [17]. Here we would like to point out the similarity of the solutions obtained within the Einstein–Maxwell–dilaton gravity and their corresponding merely classical metrics, which is not surprising.

4.3.2 String cosmology

Higher–order string cosmology is recovered if the curvature stringy corrections (4.17) are not ignored. In practise one obtains very complicated system of non–linear partial differential equations. But still there is a possibility that symmetries of a starting metric field considerably simplify the Gauss–Bonnet term.

Looking at the (4.19) and remembering that for the solution of the second class, mentioned in chapter 3 or in [18], the function γ was constant, we immediately see that for this case (4.19) vanishes. So there is no need to neglect it, and the solution of the second class from chapter 3, subject to appropriate modifications, survives even if the higher–derivative stringy corrections are taken into considerations [17]. In this sense in [17] it has been accomplished the aim to find a stringy solution which is generalization of the Gödel–type metrics. Unfortunately, with vanishing Gauss–Bonnet term, the theory loses most of stringy features.

As we have seen, the higher–order corrected actions in string cosmology usually contain terms quadratic in the Riemann tensor, which are coupled with a dilatonic or an axionic fields. Therefore it is desirable to have the expressions for their variations and in particular for the variation of the non–minimally coupled Gauss–Bonnet term. Lengthy calculations give the formula

$$\frac{\delta}{\delta g^{\mu\nu}}\left(\zeta R^2_{GB}\sqrt{-g}\right) = 4\zeta_{;\alpha\beta}R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} + 8\zeta_{;\alpha(\mu}R_{\nu)}{}^{\alpha} - 4\zeta^{;\alpha}{}_{;\alpha}R_{\mu\nu} \\ +2(\zeta^{;\alpha}{}_{;\alpha}R - 2\zeta_{;\alpha\beta}R^{\alpha\beta})g_{\mu\nu} - 2\zeta_{;\mu\nu}R\ , \qquad (4.21)$$

where ζ is a scalar field.

In a general basis the second covariant derivative of a function f is given as

$$f_{;\alpha\beta} = \mathbf{e}_\beta\left(\mathbf{e}_\alpha\right) f - \omega^{\gamma}{}_{\alpha\beta}\left(\mathbf{e}_\gamma f\right)\ ,$$

where $\mathbf{e}_\alpha$ are basis vector dual to the basis 1–forms Θ^α.

Now consider the stationary cylindrically symmetric metric (3.3) and compute the second derivatives of ζ. With the help of connection 1–forms evaluated in our orthonormal basis (3.4) they are

$$\zeta_{;\alpha\beta} = \mathrm{e}^{-\delta}\partial_r(\mathrm{e}^{-\delta}\partial_r\zeta)\delta^3_\alpha\delta^3_\beta + 2\mathrm{e}^{-\gamma}\left(\mathrm{e}^{-\delta}\partial_z\zeta - c\,\partial_r\zeta\right)\delta^2_{(\alpha}\delta^3_{\beta)} + \mathrm{e}^{-2\gamma}\partial^2_z\zeta\ \delta^2_\alpha\delta^2_\beta$$

$$-\zeta\left[f\delta^0_{(\alpha}\delta^1_{\beta)} + a\;\delta^0_\alpha\delta^0_\beta - b\;\delta^1_\alpha\delta^1_\beta - c\;\delta^2_\alpha\delta^2_\beta\right]\mathrm{e}^{-\delta}\partial_r\zeta\;. \tag{4.22}$$

Now the equations of motion should be constructed by taking a suitable *Ansatz.* Some progress has been made towards a stringy solution if (4.21) and (4.22) are taken into account. Since the work is not in any sense complete and the question arises whether there is a hope for obtaining an explicit metric, it is right time for finishing the chapter.

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Charged universes of the Gödel type with closed timelike curves

P Klepáč and J Horský

Department of Theoretical Physics and Astrophysics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic

E-mail: klepac@physics.muni.cz and horsky@physics.muni.cz

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Abstract. The aim of this paper is to examine some obtained exact solutions of the Einstein–Maxwell equations, in particular their properties from a chronological point of view. Each of our spacetimes is stationary and cylindrically symmetric and is filled with a perfect fluid that is electrically charged. There are two classes of solutions and examples of each are investigated. We give examples of the first class for both the vanishing and non-vanishing Lorentz force.

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1. Introduction

As a result of its remarkable properties the Gödel solution [1, 2, 10] has become the subject of several interesting generalizations. Banerji and Banerjee in [3] have found solutions of the Einstein–Maxwell equations for a charged perfect fluid. Generalization to the case with or without an electromagnetic field, and where the non-constant component g_{zz} or g_{tt} of the metric tensor is admitted, is possible (see [7] and references cited therein). In this paper we give a number of solutions of the Einstein–Maxwell equations with a vanishing cosmological constant which are generalizations of the metrics discussed in [3] and which involve the electromagnetic field as well as the non-constancy of g_{zz} or g_{tt}. These solutions result from two classes of exact solutions of the Einstein–Maxwell equations for a spacetime filled up with a charged perfect fluid. The subject of this paper generalizes directly from the paper given by Mitskievič and Tsalakou [4], who used the Horský–Mitskievič (HM) conjecture [5], and therefore their designations are partly preserved.

The plan of the paper is as follows. In section 2 we obtain the Einstein–Maxwell equations for the stationary and cylindrically symmetric spacetime with the vector potential (2.3). Throughout the whole paper we systematically use an orthonormal basis in which the stress–energy tensor of the perfect fluid is diagonal. In section 3 we find a general solution of the first class that corresponds to the non-constant component g_{zz} of the metric tensor. In the case of a vanishing Lorentz force we set down the inequalities implied by the energy conditions as well as the values of the Riemann tensor for this solution. Section 4 derives a second class which corresponds to the non-constancy of the g_{tt} component of the metric tensor.

2. The Einstein–Maxwell equations

In this paper, we consider a spacetime filled with a charged perfect fluid of pressure p and energy density μ. Let us choose the local coordinate system $x^{\mu} = (t, \varphi, z, r)$† of the comoving coordinates, where φ is angular, r radial and z ordinary Cartesian coordinates. We search for a metric which is stationary, cylindrically symmetric and depends on the only coordinate r, so we will restrict our attention to the case in which the spacetime admits Killing vector fields in the remaining directions, i.e. in t, φ and z directions. Since we require the metric to be invariant over simultaneous time reversion $t \to -t$ and reflection $\varphi \to -\varphi$, and over the inversion $z \to -z$, the spacetime metric tensor has the form‡

$$ds^2 = e^{2\alpha}\,(dt + f\,d\varphi)^2 - l^2\,d\varphi^2 - e^{2\gamma}\,dz^2 - e^{2\delta}\,dr^2, \tag{2.1}$$

where all metric functions as well as the pressure and the charge density depend only on the coordinate r. The function δ can achieve any value. The velocity vector is given by $u = e^{-\alpha}\partial_t$ and the acceleration of the particles of the fluid is

$$\dot{u} = \nabla_u u = \alpha' e^{-2\delta}\partial_3, \tag{2.2}$$

where the prime denotes derivative with respect to r. The expansion tensor as well the shear tensor of the spacetime with the metric (2.1) is vanishing

$$\Theta_{\mu\nu} = \tfrac{1}{2}\underset{u}{\pounds}\, h_{\mu\nu} = \tfrac{1}{2}\underset{u}{\pounds}\, g_{\mu\nu} - \dot{u}_{(\mu}u_{\nu)} = 0.$$

Finally, let the 1-form of the vector potential be expressed as (see [11])

$$A = m(r)\,d\varphi + n(r)\,dt. \tag{2.3}$$

Let us introduce an orthonormal basis with the basic 1-form as follows:

$$\Theta^{\hat{0}} = e^{\alpha}(dt + f\,d\varphi), \qquad \Theta^{\hat{1}} = l\,d\varphi,$$
$$\Theta^{\hat{2}} = e^{\gamma}\,dz, \qquad \Theta^{\hat{3}} = e^{\delta}\,dr. \tag{2.4}$$

Using (2.3) for mixed components of the 2-form of the electromagnetic field one obtains

$$F_{\mu}{}^{\nu} = l^{-2}\delta^3_{\mu}\left\{\left[n'l^2e^{-2\alpha} + (m' - fn')f\right]\delta^{\nu}_0 - (m' - fn')\delta^{\nu}_1\right\} + e^{-2\delta}\delta^{\nu}_3(n'\delta^0_{\mu} + m'\delta^1_{\mu}). \tag{2.5}$$

The Maxwell equations with sources $\delta F = -4\pi j$ give us

$$l^{-1}e^{-\alpha-\gamma-\delta}\{e^{\alpha+\gamma-\delta}l^{-1}[[n'l^2e^{-2\alpha} + (m' - fn')f]\delta^{\mu}_0 - (m' - fn')\delta^{\mu}_1]\}_{,3} = -4\pi j^{\mu}, \tag{2.6}$$

where $j(r)$ is the current density and $\delta j = 0$. We postulate that the fluid particles carry the charge and $j = \rho u$, where ρ is the charge density. This will be accomplished if

$$(m' - fn')\,e^{\alpha+\gamma-\delta} = Bl, \tag{2.7}$$

where B is constant. Equations (2.6) and (2.7) result in the following expression for the charge density:

$$4\pi\rho = -\frac{B^2}{m' - fn'}\frac{d}{dr}\left(\frac{l^2n'e^{-2\alpha}}{m' - fn'} + f\right)e^{-\alpha-2\gamma}. \tag{2.8}$$

The choice (2.3) corresponds to the formulae for the electric and the magnetic fields [4]

$$E_{\hat{r}} = n'e^{-\alpha-\delta}, \qquad B_{\hat{z}} = Be^{-\alpha-\gamma}. \tag{2.9}$$

† The coordinates will be numbered $(t, \varphi, z, r) = (0, 1, 2, 3)$.
‡ We use units with the speed of light c and the Newtonian gravity constant G set equal to one.

With the help of the constraint (2.7) the vorticity 1-form is given by

$$\omega = \tfrac{1}{2} * (u \wedge \mathrm{d}u) = \frac{Bf'}{2\,(m' - fn')}\,\mathrm{d}z.$$

The total stress–energy tensor of the spacetime T_{total} arises as a sum of the stress–energy tensor of the electromagnetic field

$$\begin{aligned} T_{\mathrm{elmag}} = (8\pi)^{-1} l^{-2} \mathrm{e}^{-2\delta} \big\{ & 2n'(m' - fn') l \mathrm{e}^{-\alpha} (\Theta^{\hat 0} \otimes \Theta^{\hat 1} + \Theta^{\hat 1} \otimes \Theta^{\hat 0}) \\ & + \left[n'^2 l^2 \mathrm{e}^{-2\alpha} + (m' - fn')^2 \right] \Theta^{\hat 0} \otimes \Theta^{\hat 0} + \left[n'^2 l^2 \mathrm{e}^{-2\alpha} + (m' - fn')^2 \right] \Theta^{\hat 1} \otimes \Theta^{\hat 1} \\ & + \left[n'^2 l^2 \mathrm{e}^{-2\alpha} - (m' - fn')^2 \right] \Theta^{\hat 2} \otimes \Theta^{\hat 2} \\ & - \left[n'^2 l^2 \mathrm{e}^{-2\alpha} - (m' - fn')^2 \right] \Theta^{\hat 3} \otimes \Theta^{\hat 3} \big\} \end{aligned} \tag{2.10}$$

and the stress–energy tensor of the perfect fluid

$$T_{\mathrm{fluid}} = \mu \Theta^{\hat 0} \otimes \Theta^{\hat 0} + p \big(\Theta^{\hat 1} \otimes \Theta^{\hat 1} + \Theta^{\hat 2} \otimes \Theta^{\hat 2} + \Theta^{\hat 3} \otimes \Theta^{\hat 3} \big). \tag{2.11}$$

The Bianchi identities reduce to a single equation of the motion of the fluid

$$p' + \alpha' (p + \mu) + \rho n' = 0. \tag{2.12}$$

The non-zero components of the Einstein tensor in the orthonormal basis (2.4) can be written in the form

$$\begin{aligned} G_{\hat t \hat\varphi} &= R_{\hat t \hat\varphi} = \tfrac{1}{2} \mathrm{e}^{-2\alpha - \gamma - \delta} \frac{\mathrm{d}}{\mathrm{d}r} \left(l^{-1} f' \mathrm{e}^{2\alpha} \mathrm{e}^{\alpha + \gamma - \delta} \right), \\ G_{\hat\varphi\hat\varphi} - G_{\hat r \hat r} &= l \mathrm{e}^{-\alpha - \gamma - \delta} \frac{\mathrm{d}}{\mathrm{d}r} \left(l^{-1} (\alpha + \gamma)' \mathrm{e}^{\alpha + \gamma - \delta} \right) - 2\alpha' \gamma' \mathrm{e}^{-2\delta}, \\ G_{\hat r \hat r} + G_{\hat z \hat z} &= \tfrac{1}{2} l^{-1} \mathrm{e}^{-\alpha - \gamma - \delta} \frac{\mathrm{d}}{\mathrm{d}r} \left[l^{-1} \mathrm{e}^{-\alpha + \gamma - \delta} \frac{\mathrm{d}}{\mathrm{d}r} \left(l^2 \mathrm{e}^{2\alpha} \right) \right], \\ G_{\hat t \hat t} - G_{\hat r \hat r} &= -\tfrac{1}{2} l^{-1} \mathrm{e}^{-\alpha - \gamma - \delta} \frac{\mathrm{d}}{\mathrm{d}r} \left[l^{-1} \mathrm{e}^{\alpha - \gamma - \delta} \frac{\mathrm{d}}{\mathrm{d}r} \left(l^2 \mathrm{e}^{2\gamma} \right) \right] + \tfrac{1}{2} l^{-2} f'^2 \mathrm{e}^{2(\alpha - \delta)}, \\ G_{\hat r \hat r} &= \left(\alpha' l' + l' \gamma' + l \gamma' \alpha' \right) l^{-1} \mathrm{e}^{-2\delta} + \tfrac{1}{4} l^{-2} f'^2 \mathrm{e}^{2(\alpha - \delta)}, \end{aligned}$$

and the Einstein–Maxwell equations $G_{\hat\mu\hat\nu} = 8\pi T_{\hat\mu\hat\nu}$ are

$$\frac{\mathrm{d}}{\mathrm{d}r} \left[\frac{\mathrm{e}^{2\alpha} f'}{m' - fn'} \right] = 4n', \tag{2.13a}$$

$$\frac{\mathrm{d}}{\mathrm{d}r} \left[\frac{(\alpha + \gamma)'}{m' - fn'} \right] - \frac{2\alpha' \gamma'}{m' - fn'} = \frac{2n'^2 \mathrm{e}^{-2\alpha}}{m' - fn'}, \tag{2.13b}$$

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}r} \left[\frac{\mathrm{e}^{-2\alpha}}{m' - fn'} \frac{\mathrm{d}}{\mathrm{d}r} \left(l^2 \mathrm{e}^{2\alpha} \right) \right] = \frac{16\pi l^2 p\, \mathrm{e}^{2\delta}}{m' - fn'}, \tag{2.13c}$$

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}r} \left[\frac{\mathrm{e}^{-2\gamma}}{m' - fn'} \frac{\mathrm{d}}{\mathrm{d}r} \left(l^2 \mathrm{e}^{2\gamma} \right) \right] = \frac{8\pi (p - \mu)}{m' - fn'} l^2 \mathrm{e}^{2\delta} + \frac{1}{2} \frac{f'^2 \mathrm{e}^{2\alpha}}{m' - fn'} - \frac{2n'^2 l^2 \mathrm{e}^{-2\alpha}}{m' - fn'}, \tag{2.13d}$$

$$(\alpha' + \gamma') \frac{\mathrm{d}l^2}{\mathrm{d}r} + 2\alpha' \gamma' l^2 = 16\pi l^2 p\, \mathrm{e}^{2\delta} - \tfrac{1}{2} f'^2 \mathrm{e}^{2\alpha} - 2n'^2 l^2 \mathrm{e}^{-2\alpha} + 2(m' - fn')^2. \tag{2.13e}$$

To summarize, we obtained the system of six equations (2.13*a*)–(2.13*e*) and (2.6) minus (2.13*e*), which is an integral of (2.12). After eliminating p from (2.13*a*)–(2.13*d*) with the help of (2.13e) one obtains the system of six equations for a total of nine unknown functions $\alpha, \gamma, \delta, f, l, \mu, \rho, m$ and n. This means that three out of the nine functions can be chosen arbitrarily. These three degrees of freedom are equivalent to introducing δ, ρ and $\mu(p)$. In the following we will consider m and n to be fixed, and moreover we impose, as in [4], an additional condition: either α or γ are constant, which is in accordance with the two classes of solutions.

3. The first class: α is constant

When α = constant the Einstein–Maxwell equations (2.13a)–(2.13d), (2.6) can be easily integrated in terms of the electric and the magnetic potential m and n. Equation (2.12) now shows us that the Lorentz force is balanced by the pressure gradient, grad $p+\rho E_{\hat{r}} = 0$, resulting in the geodesic motion of the fluid particles. We can, without loss of generality, suppose that α is zero. The metric components are

$$f e^{2n^2} = 4\int m' n\, e^{2n^2}\, dr + F, \tag{3.1a}$$

$$\gamma = \int \left[\int \frac{2n'^2}{m' - f n'}\, dr + C\right] (m' - f n')\, dr, \tag{3.1b}$$

$$l^2 e^{-2\gamma} = E - 4\int (m - f n + k)(m' - f n')\, e^{-2\gamma}\, dr. \tag{3.1c}$$

In (3.1a) C, E, F, k are constants of integration. Inserting (3.1a) into (2.13e) and (2.13d) yields for the pressure and the energy density

$$\begin{aligned} 8\pi p &= B^2\left[4n^2 + \frac{\gamma'^2 + n'^2}{(m' - f n')^2} l^2 - 2\gamma' \frac{m - f n + k}{m' - f n'} - 1\right] e^{-2\gamma}, \\ 8\pi \mu &= B^2\left[4n^2 - \frac{3\gamma'^2 + 5n'^2}{(m' - f n')^2} l^2 + 6\gamma' \frac{m - f n + k}{m' - f n'} + 1\right] e^{-2\gamma}. \end{aligned} \tag{3.2}$$

3.1. Example of the first class with a non-vanishing Lorentz force

We present explicitly the solution when $m = \tau n + \frac{2}{3}\beta n^3$, where τ and β are constants and k, F and C in (3.1a)–(3.1c) are zero. In this case (according to (2.5) the Lorentz force does not vanish) the metric can be written as follows:

$$\begin{aligned} ds^2 &= \left[dt + (2\beta n^2 + \tau - \beta)\, d\varphi\right]^2 - \tfrac{1}{3}\beta^2\left(3E\beta^{-2} e^{2n^2} - 4n^2 + 1\right) d\varphi^2 \\ &\quad - e^{2n^2}\, dz^2 - \frac{1}{B^2}\frac{3n'^2 e^{2n^2}\, dr^2}{3E\beta^{-2} e^{2n^2} - 4n^2 + 1}, \end{aligned} \tag{3.3}$$

and for the charge density (2.8), for the pressure and the energy density (3.2) one has

$$\begin{aligned} 8\pi p &= B^2\left[\frac{E}{\beta^2}(4n^2 + 1) - \tfrac{2}{3} e^{-2n^2}\right] \qquad \pi\rho = -B^2 n\left(\frac{E}{\beta^2} + \frac{1}{3} e^{-2n^2}\right), \\ 8\pi\mu &= B^2\left\{\left[\tfrac{56}{3}n^2 - \tfrac{2}{3}\right] e^{-2n^2} - \frac{E}{\beta^2}(12n^2 + 5)\right\}. \end{aligned} \tag{3.4}$$

3.2. Solution with a vanishing Lorentz force

The metric given by (3.1*a*)–(3.1*c*) is not too transparent. In the rest of this section we restrict ourselves to a purely magnetic field where the electric potential n is constant, which will be marked as $\frac{1}{4}b$, i.e. when according to (2.5) or (2.7) the Lorentz force $F^{\mu}_{\ \nu}u^{\nu}$ vanishes. The result written in terms of the magnetic potential m is† (F was gauged away)

$$\begin{aligned} \mathrm{d}s^2 &= (\mathrm{d}t + bm\,\mathrm{d}\varphi)^2 - (E\mathrm{e}^{2Cm} + \lambda m + \nu)\,\mathrm{d}\varphi^2 - \mathrm{e}^{2Cm}\,\mathrm{d}z^2 - \frac{1}{B^2}\frac{m'^2\mathrm{e}^{2Cm}}{E\mathrm{e}^{2Cm} + \lambda m + \nu}\,\mathrm{d}r^2, \\ \lambda &= \frac{4-b^2}{2C}, \qquad \nu = \frac{\lambda + 4k}{2C}. \end{aligned} \tag{3.5}$$

The charge density, pressure and the energy density are

$$\begin{aligned} 4\pi\rho &= -B^2 b\mathrm{e}^{-2Cm}, \qquad 8\pi p = B^2C^2E, \\ 8\pi\mu &= B^2\left[(b^2-2)\,\mathrm{e}^{-2Cm} - 3C^2E\right]. \end{aligned} \tag{3.6}$$

Our fluid moves geodesically, without shear or expansion‡. For m which is an increasing or decreasing function defined on $(0, \infty)$ and covering the whole range $(0, \infty)$, it can be treated as a new radial variable and one obtains the first family of the solutions in [4], if we interpret the coordinate φ as an ordinary Cartesian coordinate (i.e. one abandons the periodicity of the φ coordinate). The same will be true in section 4. The solution without the electromagnetic field ($B = 0$) was recovered by Wright [6].

3.2.1. Energy conditions. The dominant energy condition implies the inequalities

$$(b^2-2)\,\mathrm{e}^{-2Cm} - 2C^2E \geqslant 0, \tag{3.7a}$$

$$(b^2-2)\,\mathrm{e}^{-2Cm} - 4C^2E \geqslant 0. \tag{3.7b}$$

In our case the strong energy condition is satisfied if the dominant energy condition is. Altogether the energy conditions will be fulfilled if (3.7*a*) or (3.7*b*) is satisfied depending on whether E is negative or positive, respectively, or $b^2 \geqslant 2$, if E is zero.

3.2.2. Curvature. For future convenience we set down the concrete values of the Riemann curvature tensor for the metric (3.5). All the non-vanishing components of the Riemann tensor are given by

$$\begin{aligned} R_{\hat{r}\hat{z}\hat{r}\hat{z}} &= R_{\hat{\varphi}\hat{z}\hat{\varphi}\hat{z}} = \tfrac{1}{2}B^2\mathrm{e}^{-2\alpha}\left(\lambda C\mathrm{e}^{-2Cm} + 2C^2E\right), \\ R_{\hat{t}\hat{r}\hat{t}\hat{r}} &= R_{\hat{t}\hat{\varphi}\hat{t}\hat{\varphi}} = -\tfrac{1}{4}B^2b^2\mathrm{e}^{-4\alpha}\mathrm{e}^{-2Cm}, \\ R_{\hat{t}\hat{r}\hat{\varphi}\hat{r}} &= -R_{\hat{t}\hat{z}\hat{\varphi}\hat{z}} = \tfrac{1}{2}B^2bC\mathrm{e}^{-3\alpha}(E\mathrm{e}^{2Cm} + \lambda m + \nu)^{1/2}\mathrm{e}^{-2Cm}, \\ R_{\hat{\varphi}\hat{r}\hat{\varphi}\hat{r}} &= -\tfrac{1}{4}B^2\mathrm{e}^{-2\alpha}(3b^2\mathrm{e}^{-2\alpha} + 2\lambda C)\,\mathrm{e}^{-2Cm} + B^2C^2E\mathrm{e}^{-2\alpha}. \end{aligned} \tag{3.8}$$

† In fact, there is another way of deriving locally the same metric as (3.5) that introduces a new variable by rescaling the coordinate r by the definition $m(r) = m$. Then for the 2-form of the electromagnetic field $F = \mathrm{d}m \wedge \mathrm{d}\varphi$. In effect, this substitution results in inserting unity except m'.

‡ Thanks to the homogeneity of the pressure (3.6), the solution (3.5) can be reinterpreted also as a solution describing charged dust in the spacetime with the cosmological constant $\Lambda = -B^2C^2E$ (the Einstein equations in our convention are $G - \Lambda g = 8\pi T$).

3.2.3. Closed timelike curves. For the analysis of potential chronology violation we use the Carter theorem [8, 9]. In our case (3.5) the isometry group, generated by the Killing vector fields ∂_t, ∂_φ, ∂_z, is Abelian and timewise orthogonally transitive, since the hypersurfaces of transitivity given by constant r are everywhere timelike, $g^{rr} < 0$. The only exception could be the case $C = 0$, but then the chronological structure follows from the continuity and we will meet such an example in section 3.2.4. Chronology will be preserved when one is able to find constants p, q, s such that the differential form $\psi = p\,\mathrm{d}t + q\,\mathrm{d}\varphi + s\,\mathrm{d}z$ is everywhere timelike. Since the contribution of s is always negative, it can be treated as zero.

3.2.4. Example of the first class with a vanishing Lorentz force. Let the vector potential be $A = 2a^2 B\,\mathrm{sh}^2(r/2a)\,\mathrm{d}\varphi$. Here a is a length characteristic†, B is according to (2.9) a value of the magnetic field on the rotation axis. The choice of the constant of integration $E = -\nu = (2a^2B^2C^2)^{-1}$, $b^2 = 4 - 4CB^{-1} + 2(aB)^{-2}$ yields the metric

$$\mathrm{d}s^2 = \left[\mathrm{d}t + 2a\left(4a^2B^2 + 2 - 2C\right)^{1/2} m\,\mathrm{d}\varphi\right]^2 - \frac{2a^2}{C^2}\left[\mathrm{e}^{2Cm} - 2C(1-C)m - 1\right]\mathrm{d}\varphi^2$$
$$-\mathrm{e}^{2Cm}\,\mathrm{d}z^2 - \frac{2a^2C^2m'^2\mathrm{e}^{2Cm}\,\mathrm{d}r^2}{\mathrm{e}^{2Cm} - 2C(1-C)m - 1}, \qquad (3.9)$$

where, for convenience, we have denoted $2a^2BC \to C$ and $m = \mathrm{sh}^2(r/2a)$. For the physical quantities μ, p and ρ one obtains

$$8\pi\mu = 2\left(B^2 + \frac{1-C}{a^2}\right)\mathrm{e}^{-2Cm} - \frac{3}{2a^2},$$
$$2\pi\rho = -B\left(B^2 + \frac{1-C}{2a^2}\right)^{1/2}\mathrm{e}^{-2Cm}, \qquad 8\pi p = \frac{1}{2a^2}. \qquad (3.10)$$

The signature in (3.9) is always correct and the energy condition (3.7*b*) requires $C \leqslant 0$. From equations (3.8) we can see that for negative C the physical singularity occurs when $r \to \infty$. From (3.9) one can also see that no event horizon is present for any finite value of r. In the limit $C = 0$ we get the Banerji–Banerjee solution (formulae (11) and (12) in the alternative interpretation) in [3] which is spacetime homogeneous and singularity free. So we can interpret C as an indicator of the difference of the spacetime from the spacetime homogeneity.

In fact, the choice $m = \mathrm{sh}^2(r/2a)$ represents the family of models in which m is an increasing or decreasing function with domain $(0, \infty)$ and image $(0, \infty)$. Each such choice after the introduction of the new variable $m = m(r)$ leads to equation (3.9). From this point of view it is obvious that, for example, the choice $m \propto r^{-1}$ has the same physical content as $m = \mathrm{sh}^2(r/2a)$. The same will be true in subsection 4.3.

The condition $\psi_\alpha\psi^\alpha \geqslant 0$ of the section 3.2.3 reads

$$\frac{1}{2C^2}\left[\mathrm{e}^{2Cm} - 2C(1-C)m - 1\right] - \left[\left(4a^2B^2 + 2 - 2C\right)^{1/2} m - \frac{q}{2ap}\right]^2 \geqslant 0. \qquad (3.11)$$

Because q must be zero in order not to predominate in (3.11) for small r, the condition for the non-existence of the closed timelike curves (CTC) has the familiar form $g_{\varphi\varphi} \leqslant 0$ (or $g_{\varphi\varphi} < 0$ for the non-existence of the closed causal curves), and we see that CTC will always appear for sufficiently large r.

As a result of the absence of any event horizons, for any two spacetime points p and q, one has $p \gg q$ and simultaneously $q \gg p$ [10]. In particular, $I^+(p) = I^-(p) = J^+(p) =$

† With respect to the previous footnote, $a^2 = -(2\Lambda)^{-1}$ holds in the alternative reinterpretation.

Table 1. Numerical dates that relate the quantity of the magnetic field B on the rotation axis to the radius R of the first null circle (t, z, r = constant), and to the cosmological constant Λ in the alternative interpretation. For simplicity, dates are given only for $C = 0$. The matter density used is 10^{-26} kg m^{-3}, which corresponds to a period of rotation of 7×10^{10} years.

B (10^3 G)	0.0	3.7	4.8	6.7	8.2	10.5
Λ (10^{-53} m^{-2})	−9.31	−8.20	−7.45	−5.59	−3.73	−0.28
R (10^8 ly)	137	132	129	123	118	110

$J^-(p) = M$. Although a CTC passes through every point, each such curve must inevitable cross the region $r > R$, where R is determined as a root of (3.11). The CTC we are interested in are non-trivial [8]. Moreover, because the Carter theorem can be applied independently of whether or not φ is periodic, one can see that CTC occur even if we interpret φ as the ordinary Cartesian coordinate. (The chronology could be preserved if we took only the part of (3.9) confined to the region $r < R$ and tried to match it to some other chronologically well behaved solution.)

The magnetic field always has a positive influence on the chronology violation, that is the larger the value of the magnetic field on the rotation axis the closer to it will be the 'first' null circle followed by CTC (see table 1).

Similarly, another Banerjee–Banerji solution (expression (15) in [3]) or the Som and Raychaudhuri solution [11] can be obtained by a suitable choice of the constant of integration.

4. The second class of solutions: γ is constant

The solution with constant γ will be referred to as second class and here we have restricted ourselves to the case when the Lorentz force vanishes, with the electric potential $n = b/4$. Equation (3.1c) remains valid in this case too. γ can be set to zero and the solution of the Einstein–Maxwell equations (2.13a)–(2.13d) and (2.6) can be expressed in the form

$$f = -\frac{b}{2C}\mathrm{e}^{-2Cm} + H, \qquad \mathrm{e}^{2\alpha} = \mathrm{e}^{2Cm},$$
$$l^2 = \frac{b^2}{4C^2}\mathrm{e}^{-2Cm} - 2m^2 + Dm + E, \tag{4.1}$$

with constants of integration C, H and $D = Hb - 4k$. For the energy density, the pressure and the charge density one has

$$16\pi\mu = B^2\mathrm{e}^{-2Cm}[CD + 2(1 - 2Cm)],$$
$$16\pi p = B^2\mathrm{e}^{-2Cm}[CD - 2(1 + 2Cm)], \qquad 4\pi\rho = -B^2 b\mathrm{e}^{-3Cm}. \tag{4.2}$$

As a result of the inhomogeneity of the pressure (4.2), the fluid already does not move along geodesic lines. Its acceleration (2.2) is

$$\nabla_u u = Cm'\mathrm{e}^{-2\delta}\partial_3.$$

In contrast to the corresponding solution of the first class (3.9), because of the inhomogeneity of the pressure, the metrics of the second class cannot be reinterpreted as describing dust with the non-zero cosmological constant.

4.1. Energy conditions

In the case of the second class (4.1) and (4.2) the energy conditions will all be fulfilled if the following inequality holds:

$$CD \geqslant 4Cm. \tag{4.3}$$

4.2. Curvature

The non-zero components of the Riemann tensor are given by

$$\begin{aligned} R_{\hat{t}\hat{r}\hat{t}\hat{r}} &= R_{\hat{t}\hat{\varphi}\hat{t}\hat{\varphi}} = B^2 C \mathrm{e}^{-2\alpha}\left(2m - \tfrac{1}{2}D\right)\mathrm{e}^{-2Cm}, \\ R_{\hat{r}\hat{\varphi}\hat{r}\hat{\varphi}} &= B^2 \mathrm{e}^{-2\alpha}\left[C\left(2m - \tfrac{1}{2}D\right) - 2\right]\mathrm{e}^{-2Cm}. \end{aligned} \tag{4.4}$$

4.3. Example of the second class of solutions

This special solution corresponds to that discussed in section 3.2.4 in that when C goes to zero, we obtain the same solution as in section 3.2.4 when C goes to zero, i.e. (11) and (12) in [3]. Let us choose $\mathrm{A} = 2a^2 B\,\mathrm{sh}^2(r/2a)\,\mathrm{d}\varphi$ and the constants of integration $b^2 = 4 + 2(aB)^{-2}$, $D = 2B^{-1} + 2C^{-1} + \left(a^2B^2C\right)^{-1}$, $E = -b^2(2C)^{-2}$, $H = b(2C)^{-1}$. Components of the metric tensor read (denoting $2a^2BC \to C$ and $m = \mathrm{sh}^2(r/2a)$)

$$\begin{aligned} &\mathrm{e}^{2\alpha} = \mathrm{e}^{2Cm}, \qquad f = a\frac{\left(4a^2B^2+2\right)^{1/2}}{C}\left(1 - \mathrm{e}^{-2Cm}\right), \\ &l^2 = 4a^2\left[\frac{2a^2B^2+1}{2C^2}\mathrm{e}^{-2Cm} - 2a^2B^2m^2 + \frac{(2a^2B^2+1+C)m}{C} - \frac{2a^2B^2+1}{2C^2}\right]. \end{aligned} \tag{4.5}$$

The energy density, the pressure and the charge density are given, respectively, by

$$\begin{aligned} 8\pi\mu &= \left(2B^2 + \frac{1+C}{2a^2} - 2B^2Cm\right)\mathrm{e}^{-2Cm}, \\ 8\pi p &= \left(\frac{1+C}{2a^2} - 2B^2Cm\right)\mathrm{e}^{-2Cm}, \\ 2\pi\rho &= -B\left(B^2 + \frac{1}{2a^2}\right)^{1/2}\mathrm{e}^{-3Cm}. \end{aligned} \tag{4.6}$$

It can be seen from the form of metrics (4.5) that the signature is correct for $C \leqslant 0$ and, in the case when B is zero, then for every C. The absence of the event horizons for any finite value of r is apparent from the form of the metric (4.5). From analysis of the energy conditions (4.3) it follows that these will be fulfilled for intervals $-(2a^2B^2+1) \leqslant C \leqslant 0$, and if the electromagnetic field vanishes for $C \geqslant -1$. One can convince oneself that CTC will occur if $-(2a^2B^2+1) \leqslant C \leqslant 0$ when $B \neq 0$, and $C > -1$ when $B = 0$. Every two points of the spacetime can be connected to each other by both a future- and past-directed timelike curve which is non-trivial, so that the spacetime is totally viscous [8]. Formulae (4.4) show that the physical singularity occurs for $C < 0$ when r tends to infinity. The only exception is the case with the vanishing electromagnetic field ($B = 0$) and $C = -1$, which is after transformation $\mathrm{e}^{-2m} = 1 - \omega^2u^2$, with $\omega^{-2} = 2a^2$, the Minkowski spacetime in the rotating cylindrical coordinates $\varphi \to \varphi - \omega t$.

5. Conclusion

In this paper we have found metrics of the Gödel type, i.e. metrics of which the Gödel spacetime is a special example. From the requirement of the stationarity and cylindrical symmetry and from the assumption that the spacetime is filled with a perfect fluid that is charged, we have obtained the system of ordinary differential equations (2.13*a*)–(2.13*d*) plus (2.6), that was solved in terms of the functions m and n in two cases: the first class (3.1*a*)–(3.1*c*), (3.2) with constant α and the second class (4.1), (4.2) with constant γ (however, we gave the solution of the second class only for the vanishing Lorentz force). These two general classes are both shear-free. A fluid of the first class moves geodesically and its velocity is the Killing vector field. This is not true in the second class owing to the inhomogeneity of the pressure (4.2). We have shown an explicit special solution of the first class for the non-vanishing Lorentz force (formulae (3.3) and (3.4)), but without detailed analysis, and we have discussed in some detail the example of the first class (section 3.2.4) and the corresponding example of the second class (section 4.3) with the vanishing Lorentz force, that are generalizations of the solution given in [3]. These metrics with three parameters (the length characteristic a, the value of the magnetic field on the rotation axis B, and C determining the difference of the solutions from the homogeneity) contain CTC at least for some interval of the values of C. It turns out that the magnetic field has a positive influence on the appearance of the non-trivial chronology violation (in fact with a sufficiently large magnetic field we can always ensure the chronology violation).

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CHARGED PERFECT FLUID AND SCALAR FIELD COUPLED TO GRAVITY

P. KLEPÁČ[1] and J. HORSKÝ[2]

Institute of Theoretical Physics and Astrophysics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic

Charged perfect fluid with vanishing Lorentz force and massless scalar field is studied in the case of stationary cylindrically symmetric spacetime. The scalar field can depend both on radial and longitudinal coordinates. Solutions are found and classified according to scalar field gradient and magnetic field relationship. Their physical and geometrical properties are examined and discussion of particular cases, directly generalizing Gödel-type spacetimes, is presented.

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1 Introduction

Since Gödel original work [1] in 1949 the rotating charged perfect fluid cosmology has become an attractive subject for many scientists. These cosmological models, mostly restricted to Lorentz force-free cases, are remarkable as they exhibit the interplay of rotation, gravitation and electromagnetism. Wright [2] constructed inhomogeneous class of electrically neutral perfect fluid solutions. Banerjee and Banerji [3] gave electromagnetic Gödel-type solution, considering longitudinal magnetic field parallel to rotation axis. Solution proposed by Som and Raychaudhuri [4] is electromagnetic analog of van Stockum solution [5]. Bonnor [6] obtained solutions for axially symmetric dust spacetimes both for non-vanishing and vanishing Lorentz force, the latter being general one (if rotation is rigid). In ninetieth Mitskiévič and Tsalakou [7] found inhomogeneous charged G_3 spacetimes using Horský-Mitskievič conjecture [8] and their results were generalized to an arbitrary magnetic potential by Upornikov [9]. Physical interpretation of Lorentz force-free charged fluids in general relativity can be found in [10]. Recently, Klepáč and Horský submitted G_3 spacetimes for generally non-vanishing Lorentz force [11].

The reason why to study the coupling of the charged perfect fluid and the scalar field to gravity relies on the fact that in modern unified theories such as string and superstring theories ([12, 13]), there appear effective actions reproducing, at the 1-loop level, the Einstein field equations enriched in four dimensions by a contribution of one or more scalar fields (dilaton, axion, etc.). Therefore it is natural, as a first step, to search for solutions of the Einstein equations describing the coupling of a

[1]E-mail: klepac@physics.muni.cz
[2]E-mail: horsky@physics.muni.cz

charged perfect fluid and a dilaton to gravity. Then the fluid can be viewed as an very rough approximation of the fermionic matter in the theory.

The paper is organized as follows. In section 2 basic assumptions and equations are introduced and in section 3 main solutions are derived. Section 4 specializes general results of section 3, intending to clarify their connection with Gödel-type metrics. There is the remark on scalar field behaviour and brief conclusion in 5.

2 The Einstein-Maxwell equations

We are searching for stationary cylindrically symmetric cosmological models with charged perfect fluid and scalar field coupled to gravity. Let us introduce locally Lorentz coframe $\mathrm{d}s^2 = \eta_{\alpha\beta}\Theta^{\hat\alpha} \otimes \Theta^{\hat\beta}$ with basis 1-forms

$$\begin{aligned} \Theta^{\hat 0} &= \mathrm{d}t + f\,\mathrm{d}\varphi\ , & \Theta^{\hat 1} &= l\,\mathrm{d}\varphi\ , \\ \Theta^{\hat 2} &= \mathrm{e}^{\gamma}\,\mathrm{d}z\ , & \Theta^{\hat 3} &= \mathrm{e}^{\delta}\,\mathrm{d}r\ , \end{aligned} \tag{2.1}$$

where $(t,\varphi,z,r) = (x^0,x^1,x^2,x^3)$ are local cylindrical coordinates on a manifold, *adapted* to the Killing vector fields ∂_t, ∂_φ, ∂_z, and f, l, γ, δ are functions of r alone. If one assumes that a spacetime we are looking for is filled with the perfect fluid of pressure $p(r)$ and energy density $\mu(r)$ moving in its rest frame with velocity $u = \Theta^{\hat 0}$, then the motion of the fluid is characterized by geodesic rigid rotation around z axis, $\dot u = 0$, $\Theta = \sigma = 0$. Thus velocity is the Killing vector field.

Since in this paper we concentrate on cases with vanishing Lorentz force, in the fluid comoving frame there is purely magnetic field present. Excluding cylindrically symmetric spacetimes with currents parallel to z-axis, due to symmetry only

$$B_{\hat r} = F_{\varphi z}(r)l^{-1}\mathrm{e}^{-\gamma}\ , \qquad B_{\hat z} = F_{r\varphi}(r)l^{-1}\mathrm{e}^{-\delta} \tag{2.2}$$

components of the magnetic field survive. Note that here we consider only r dependent electromagnetic field. In order to satisfy condition $\mathrm{d}F = 0$ it follows that $F_{\varphi z}$ is constant. If fluid particles are charge carriers, the Maxwell equations with sources $- * \mathrm{d} * F = 4\pi j = 4\pi\rho u$ give expressions for z-component of the magnetic field and for the invariant charge density ρ. Denoting for future convenience $M = l\mathrm{e}^{-\gamma+\delta}$ one has

$$F_{r\varphi} = Bl\mathrm{e}^{-\gamma+\delta} \equiv BM\ , \qquad 4\pi\rho = -\frac{B}{M}\frac{\mathrm{d}f}{\mathrm{d}r}\,\mathrm{e}^{-2\gamma}\ , \tag{2.3}$$

B being constant characterizing longitudinal component of the magnetic field.

Finally we incorporate massless scalar field ϕ into the theory. Owing to the symmetry of the problem, ϕ is, in general, assumed to be function of r as well as of z. Stress-energy tensor associated with the scalar field is taken in the usual way

$$8\pi T_{\mathrm{scal}} = \left[(\mathbf{e}_{\hat\mu}\phi)\ (\mathbf{e}_{\hat\nu}\phi) - \frac{1}{2}\eta_{\mu\nu}\,(\nabla\phi)^2\right]\Theta^{\hat\mu}\otimes\Theta^{\hat\nu}\ , \tag{2.4}$$

where $\mathbf{e}_{\hat{\mu}}$ are basis vector fields dual to basis 1-forms (2.1)

$$\begin{aligned} \mathbf{e}_{\hat{0}} &= \partial_t \ , & \mathbf{e}_{\hat{1}} &= l^{-1}\left(\partial_\varphi - f\partial_t\right) \ , \\ \mathbf{e}_{\hat{2}} &= \mathrm{e}^{-\gamma}\partial_z \ , & \mathbf{e}_{\hat{3}} &= \mathrm{e}^{-\delta}\partial_r \ . \end{aligned} \tag{2.5}$$

Appropriate Einstein-Maxwell equations read (with $c = G = 1$)

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{M}\frac{\mathrm{d}f}{\mathrm{d}r}\right] = 0 \ , \tag{2.6a}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{M}\frac{\mathrm{d}\gamma}{\mathrm{d}r}\right] = -\frac{1}{M}\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{2M}{l^4}F_{\varphi z}^2 \ , \tag{2.6b}$$

$$\frac{M}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{M}\frac{\mathrm{d}l^2}{\mathrm{d}r}\right] = 16\pi l^2 p\,\mathrm{e}^{2\delta} \ , \tag{2.6c}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{e}^{-2\gamma}}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left(l^2\mathrm{e}^{2\gamma}\right)\right] = {} & \frac{16\pi(p-\mu)}{M}l^2\mathrm{e}^{2\delta} + \frac{1}{M}\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 - 2M\left(\frac{\partial\phi}{\partial z}\right)^2 \\ & - 4F_{\varphi z}^2 l^{-1}\mathrm{e}^{-\gamma+\delta} \ , \end{aligned} \tag{2.6d}$$

$$\begin{aligned} \frac{\mathrm{d}\gamma}{\mathrm{d}r}\frac{\mathrm{d}\,l^2}{\mathrm{d}r} = {} & 16\pi l^2 p\,\mathrm{e}^{2\delta} - \frac{1}{2}\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 + 2B^2M^2 - 2F_{\varphi z}^2\mathrm{e}^{2\delta-2\gamma} \\ & + l^2\left(\frac{\partial\phi}{\partial r}\right)^2 - M^2\left(\frac{\partial\phi}{\partial r}\right)^2 \ , \end{aligned} \tag{2.6e}$$

$$\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial z} = 2BMl^{-2}F_{\varphi z} \ , \tag{2.6f}$$

with ϕ subject to scalar equation of motion

$$\frac{1}{M}\frac{\partial}{\partial r}\left(\frac{l^2}{M}\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial z^2} = 0 \, . \tag{2.7}$$

Equation (2.6e) is integral of the Bianchi identity that claims constant pressure p (by virtue of geodesic motion) if (2.7) holds, which in turn appears to be satisfied identically if the Einstein equations are. Therefore after eliminating p from remaining Einstein equations one has altogether seven independent equations (2.6a)-(2.6d), (2.6f) and two non-trivial Maxwell equations for eight unknown functions f, l, γ, δ, μ, ρ, $F_{\varphi r}$ and ϕ. One degree of freedom corresponds to the possibility of choosing arbitrarily the scale function δ. Equation (2.6a) gives $f = 2\Omega m + D$, where $m = \int M\,\mathrm{d}r$. In case of no radial magnetic field m represents (non-uniquely) angular component of magnetic potential (i.e. vector potential is $A = Bm\,\mathrm{d}\varphi$). Constant Ω is the rate of the rigid rotation of matter around z-axis, with vorticity covector $\omega = \frac{1}{2} * (u \wedge \mathrm{d}u) = \frac{1}{2}\frac{f'}{M}\,\mathrm{d}z = \Omega\,\mathrm{d}z$, with prime denoting derivative with respect to r. Unimportant constant D will be omitted further.

Because of the independence of r and z-coordinates it follows from (2.6d), (2.6b) and (2.6f) that ϕ is expressed like

$$\phi = \phi_0 + \phi_1 z + \phi_2 \int \frac{M}{l^2}\,\mathrm{d}r\,, \qquad \phi_2 = \frac{2BF_{\varphi z}}{\phi_1}\,, \tag{2.8}$$

where ϕ_0, ϕ_1 are constants and the latter equality being valid provided that $\phi_1 \neq 0$.

By integrating of combination of (2.6c), (2.6e) and (2.6b), and excluding γ, one obtains a non-linear equation relating l^2 to M

$$\frac{l^4}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\frac{1}{M}\frac{\mathrm{d}l^2}{\mathrm{d}r} + \left(4B^2 - 2\phi_1^2 - 4\Omega^2\right)m + 4k\right)l^{-2}\right] = 4F_{\varphi z}^2 - 2\phi_2^2\,. \tag{2.9}$$

In (2.9) an integration constant is denoted $4k$ in conformity with [11].

We proceed further by splitting solutions into two groups according to whether right-hand side of (2.9) vanishes or not.

3 Exact solutions

Case I: $\phi_2^2 = 2F_{\varphi z}^2$.

Radial part of scalar field gradient balances the radial component of magnetic field. This case will be itemized in two subcases.

(a) If $F_{z\varphi} \neq 0$ then solution of the Einstein-Maxwell system reads ($P = 8\pi p$)

$$\mathrm{d}s^2 = \left(\mathrm{d}t + \frac{2\Omega}{C}\gamma\,\mathrm{d}\varphi\right)^2 - \left(\frac{P}{C^2}\mathrm{e}^{2\gamma} - \frac{2\Omega^2}{C^2}\gamma + \nu\right)\mathrm{d}\varphi^2 - \mathrm{e}^{2\gamma}\,\mathrm{d}z^2 - \frac{\gamma'^2\mathrm{e}^{2\gamma}\,\mathrm{d}r^2}{P\mathrm{e}^{2\gamma} - 2\Omega^2\gamma + C^2\nu}\,, \tag{3.1}$$

with C, ν as integration constants and γ an arbitrary non-constant twice differentiable function of r. Metric (3.1) is that given by Wright more then thirty years ago [2]. In his work, Wright considered a dust spacetime with non-vanishing cosmological constant.

Thus we have succeeded in finding an alternative source for the Wright metric. In this way ambiguity in sources for the Gödel metric disscussed in [14] is now generalized to ambiguity in sources for the Wright metric. Here we have come across with non-zero charge and energy densities given by

$$2\pi\rho = -B\Omega\mathrm{e}^{-2\gamma}\,,\ \mu = \frac{1}{4\pi}\left(2\Omega^2 - B^2 - l^{-2}F_{\varphi z}^2\right)\mathrm{e}^{-2\gamma} - 3p\,, \tag{3.2}$$

and scalar field ϕ, that is defined by (2.8), and cannot be determinated explicitly. In second equation of (3.2) the rotation, represented by vorticity scalar $\omega^2 = |\omega_\nu\omega^\nu| = \Omega^2\mathrm{e}^{-2\gamma}$, is compensated by addition of magnetic energy density and "specific" mass density $\mu + 3p$ (it is interesting to compare (3.2) with equation (6.2) in Bonnor [6]).

The integration constants in (3.1) are to be chosen properly in order to ensure Lorentzian signature, energy conditions and even regularity at the origin, if desired

[15]. In order to fulfil the energy conditions, $\omega^2 \geq P$ or $2\omega^2 \geq P$ depending on whether the pressure is positive or negative, respectively.

The qualitative decription of the radial magnetic and scalar fields behaviour in the vicinity of the rotation axis $r = 0$ can be carried out as follows. Equation (3.1) restricted to the (r, φ) subspace for small values of r reads

$$\mathrm{d}s_2^2 = C^{-2} l^{-2} (\,\mathrm{d}e^{\gamma})^2 + l^2\,\mathrm{d}\varphi^2\ , \tag{3.3}$$

and is regular for $l = 0$ provided $e^{\gamma} \sim K \pm Cl^2/2$, K is constant (see exact values of the integration constants given below). Then from (2.8) one has

$$\phi = \phi_0 + \phi_1 z + \phi_2 C^{-1} \int l^{-2}\,\mathrm{d}\gamma = \phi_0 + \phi_1 z \pm \phi_2 K^{-1} \ln \frac{2l}{|Cl^2 \pm 2K|^{1/2}}\ . \tag{3.4}$$

In this way the radial part scalar field behaves as $\phi \sim \phi_0 + \phi_1 z \pm \frac{\phi_2}{K} \ln l$, which is the singular solution of the Laplace equation of motion for line of a scalar charge at $l = 0$. Divergences in z-infinities represent additional sources of scalar charge and they are common for scalar fields in Gödel-type metrics both in classical [14] as well as in the string theories [12, 13] (solution (5.1) below is singular for the same reason). For small r the scalar field gradient radial part goes as $\frac{\partial \phi}{\partial r} \sim r^{-1}$. On the other hand the radially pointing magnetic field $B_{\hat{r}}$ defined in (2.2) is also singular on the $l = 0$ line, that acts like a linear source of the radial magnetic field (which is the same as in the classical theory), and can be interpreted as a "line of magnetic charge". In the neighbourhood of the rotation axis also $B_{\hat{r}} \sim r^{-1}$ and divergences produced by scalar field gradient along with that produced by radial magnetic field are canceled mutually keeping the total stress-energy tensor regular on the rotation axis.

(b) Subcase $F_{z\varphi} = \phi_2 = 0$ yields metric

$$\mathrm{d}s^2 = \left(\mathrm{d}t + \frac{2\Omega}{C}\gamma\,\mathrm{d}\varphi\right)^2 - \left(\frac{P}{C^2}e^{2\gamma} + \frac{\lambda}{C}\gamma + \nu\right)\mathrm{d}\varphi^2 - e^{2\gamma}\,\mathrm{d}z^2 - \frac{\gamma'^2 e^{2\gamma}\,\mathrm{d}r^2}{Pe^{2\gamma} + C\lambda\,\gamma + C^2\nu}\ ,$$

$$\lambda = \frac{2B^2 - 2\Omega^2 - \phi_1^2}{C}\ ,\ \nu = \frac{\lambda + 4k}{2C}\ . \tag{3.5}$$

The charge density, pressure and the energy density are obtained from (3.2) for $F_{z\varphi} = 0$. Scalar field becomes equal to $\phi = \phi_0 + \phi_1 z$ and does not enter the energy density, which could be expected because of the lack of interaction between charged fluid and scalar field. If both ϕ_1 and ϕ_2 are zero, (3.5) convertes to Lorentz force-free solution presented in [11, 9].

If one requires (3.1) or (3.5) to be cylindrically symmetric and regular at the origin, one has to impose axial symmetry condition

$$X \equiv \partial_\varphi \cdot \partial_\varphi = g_{\varphi\varphi} \propto \mathrm{O}(r^2) \tag{3.6}$$

as $r \to +0$, and elementary flatness condition

$$\frac{X_{,\alpha} X_{,\beta} g^{\alpha\beta}}{4X} = \frac{g^2_{\varphi\varphi,r}}{4 g_{\varphi\varphi} g_{rr}} \to 1 \tag{3.7}$$

when $r \to +0$ ([5, 15]). Assuming for simplicity $\gamma \propto r^2$, when r tends to zero (in practise $\gamma \propto \frac{C}{2}r^2$), the conditions (3.6) and (3.7) applied on (3.5) show following relations

$$\frac{1}{2}C\lambda + P = C \ , \ C^2\nu + P = 0 \ . \tag{3.8}$$

The metric (3.5) turns out to be algebraically general, except for the hypersurfaces (which could be of even infinite number for given integration constants, e.g. for periodic γ) on which

$$\left|4\Omega^2 + C\lambda\right| = 4\left|\Omega C l\right| \ , \tag{3.9}$$

where it belongs to the type II.

The energy conditions imposed on total stress-energy tensor demand inequalities given in the table 1.

Table 1: The energy conditions implied by values of ϕ_1^2 shown on upper line and p in first column.

	$\phi_1^2 \le 2B^2$	$\phi_1^2 \ge 2B^2$
$p \le 0$	$4\Omega^2 + \phi_1^2 - 2B^2 \ge 2P\mathrm{e}^{2\gamma}$	$2\omega^2 \ge P$
$p \ge 0$	$4\Omega^2 + \phi_1^2 - 2B^2 \ge 4P\mathrm{e}^{2\gamma}$	$\omega^2 \ge P$

Case II: $\phi_2^2 \neq 2F_{\varphi z}^2$.

We take non-trivial solution of (2.9) describing charged dust and scalar field distribution. The metric, written down in terms of an arbitrary non-constant twice differentiable function m, becomes

$$\mathrm{d}s^2 = (\,\mathrm{d}t + 2\Omega m\,\mathrm{d}\varphi)^2 - (\lambda m + \tilde{\nu})\,\mathrm{d}\varphi^2 - (\lambda m + \tilde{\nu})^\sigma\, b^{-\sigma}\mathrm{e}^{2Cm}\left(\mathrm{d}z^2 + \frac{m'^2\,\mathrm{d}r^2}{\lambda m + \tilde{\nu}}\right) \ , \tag{3.10}$$

and the charge and energy density are found to be

$$\begin{aligned} 2\pi\rho &= -B\Omega\,(\lambda m + \tilde{\nu})^{-\sigma}\, b^\sigma \mathrm{e}^{-2Cm} \\ 4\pi\mu &= \left(2\Omega^2 - B^2 - l^{-2}F_{\varphi z}^2\right)(\lambda m + \tilde{\nu})^{-\sigma}\, b^\sigma \mathrm{e}^{-2Cm} \ . \end{aligned} \tag{3.11}$$

Constants σ and $\tilde{\nu}$ are defined by

$$\sigma = \frac{2\phi_2^2 - 4F_{\varphi z}^2}{\lambda^2} \ , \ \tilde{\nu} = \nu - \frac{\lambda}{2C}\sigma \ ,$$

with constant b being of square length dimension.

Notice that if $B_{\hat{r}}$ is everywhere zero the charge to mass density ratio has a fixed value through the spacetime. This result is consistent with theorem in Bonnor [6].

The scalar field ϕ is given by (2.8) leading to

$$\mathrm{e}^{\phi} = (\lambda m + \tilde{\nu})^{\frac{\phi_2}{\lambda}} \, \mathrm{e}^{\phi_0 + \phi_1 z} \ , \tag{3.12}$$

and b is involved in constant ϕ_0.

Next we turn to the axial symmetry of (3.10). Assuming $m \propto r^2$ as $r \to 0$, the conditions (3.6) and (3.7) read

$$\tilde{\nu} = 0 \ , \qquad \frac{\lambda^2}{4} \left(\frac{b}{\tilde{\nu}} \right)^{\sigma} = 1 \ ,$$

from which we see that the solution (3.10) cannot represent cylindrically symmetric spacetime regular at the origin unless $\sigma = 0$ when it coincides with dust version of (3.1). As a matter of fact, there are two possibilities to retain (3.10) physically admissible. Either relax the elementary flatness on the symmetry axis, and such spacetimes are sometimes accepted ([15]), or abandon φ-coordinate periodicity.

Putting $m = \frac{\lambda}{4} r^2$, $\tilde{\nu} = 0$ and carrying out coordinate transformation $\tilde{\varphi} = \frac{\lambda}{2}\varphi$, one can establish (3.10) in more convenient form

$$\mathrm{d}s^2 = \left(\mathrm{d}t + \Omega r^2 \, \mathrm{d}\tilde{\varphi} \right)^2 - r^2 \, \mathrm{d}\tilde{\varphi}^2 - \left(\frac{\lambda^2}{4b} \right)^{\sigma} r^{2\sigma} \mathrm{e}^{\frac{C\lambda}{2} r^2} \left(\mathrm{d}z^2 + \mathrm{d}r^2 \right) \ , \tag{3.13}$$

which strongly resembles Som and Raychaudhuri solution [4], but differs from it by g_{zz} component, which gets more complicated.
The spacetime (3.10) is again algebraically general, except on hypersurfaces

$$\left| 4\Omega^2 + C\lambda + \frac{\sigma}{2} \left(\frac{\lambda}{l} \right)^2 \right| = \left| \frac{2\Omega l}{\lambda} \left[2C\lambda + \sigma \left(\frac{\lambda}{l} \right)^2 \right] \right| \ , \tag{3.14}$$

where it is of the type II.

We finish this section by setting down explicit expressions when the energy conditions are satisfied. Following inequalities must hold

$$2\Omega^2 \geq F_{\varphi z}^2 l^{-2} \left(1 - \frac{2B^2}{\phi_1^2} \right) \qquad \text{if } \phi_1^2 \geq 2B^2 \ ,$$
$$4\Omega^2 + \phi_1^2 \geq 2B^2 \qquad \text{if } \phi_1^2 \leq 2B^2 \ .$$

4 Connection with Gödel-type metrics

Solution (3.5), which will be presented in this section, includes a number of known metrics as its particular cases. From set of the solutions of some physical interest (i.e. satisfying (3.8) and the energy conditions) we present here such that recover Gödel-type metrics [14] [3] in limiting process $C \to 0$.

[3] In [14] there is source-free electromagnetic field, which does not affect form of the metric.

A transition from general (3.5) to important special cases can take one of the following three forms:

(i) With a choice $2P = -C^2\nu = \alpha^2 > 0,\ \gamma = 2C\alpha^{-2}\,\mathrm{sh}^2\left(\frac{\alpha r}{2}\right) \equiv \tilde{C}\,\mathrm{sh}^2\left(\frac{\alpha r}{2}\right)$, α is of inverse length dimension constant, one gets the metric

$$\begin{aligned} \mathrm{d}s^2 &= \left[\mathrm{d}t + \tfrac{4\Omega}{\alpha^2}\,\mathrm{sh}^2\left(\tfrac{\alpha r}{2}\right)\mathrm{d}\varphi\right]^2 - \mathrm{e}^{2\tilde{C}\,\mathrm{sh}^2\left(\frac{\alpha r}{2}\right)}\mathrm{d}z^2 \\ &- \tfrac{2}{\tilde{C}^2\alpha^2}\left[\mathrm{e}^{2\tilde{C}\,\mathrm{sh}^2\left(\frac{\alpha r}{2}\right)} - 2\tilde{C}(1-\tilde{C})\,\mathrm{sh}^2\left(\tfrac{\alpha r}{2}\right) - 1\right]\mathrm{d}\varphi^2 \\ &- \frac{\tilde{C}^2}{2}\frac{\mathrm{sh}^2(\alpha r)\mathrm{e}^{2\tilde{C}\,\mathrm{sh}^2\left(\frac{\alpha r}{2}\right)}\,\mathrm{d}r^2}{\mathrm{e}^{2\tilde{C}\,\mathrm{sh}^2\left(\frac{\alpha r}{2}\right)} - 2\tilde{C}(1-\tilde{C})\,\mathrm{sh}^2\left(\frac{\alpha r}{2}\right) - 1}\ , \end{aligned} \tag{4.1}$$

where $2\Omega^2 = 2B^2 + \alpha^2\left(1-\tilde{C}\right) - \phi_1^2$, $\tilde{C} \leq 0$ and $2B^2 - \tilde{C}\alpha^2 \geq \phi_1^2$.

(ii) Purely imaginary substitution $\alpha \to i\alpha$ in previous case gives the metric

$$\begin{aligned} \mathrm{d}s^2 &= \left[\mathrm{d}t + \tfrac{4\Omega}{\alpha^2}\sin^2\left(\tfrac{\alpha r}{2}\right)\mathrm{d}\varphi\right]^2 - \mathrm{e}^{2\tilde{C}\sin^2\left(\frac{\alpha r}{2}\right)}\mathrm{d}z^2 \\ &- \tfrac{2}{\tilde{C}^2\alpha^2}\left[1 + 2\tilde{C}(1+\tilde{C})\sin^2\left(\tfrac{\alpha r}{2}\right) - \mathrm{e}^{2\tilde{C}\sin^2\left(\frac{\alpha r}{2}\right)}\right]\mathrm{d}\varphi^2 \\ &- \frac{\tilde{C}^2}{2}\frac{\sin^2(\alpha r)\mathrm{e}^{2\tilde{C}\sin^2\left(\frac{\alpha r}{2}\right)}\,\mathrm{d}r^2}{1 + 2\tilde{C}(1+\tilde{C})\sin^2\left(\frac{\alpha r}{2}\right) - \mathrm{e}^{2\tilde{C}\sin^2\left(\frac{\alpha r}{2}\right)}}\ , \end{aligned} \tag{4.2}$$

together with $2\Omega^2 = 2B^2 - \alpha^2\left(1+\tilde{C}\right) - \phi_1^2$, $\tilde{C} \leq 0$ and let $2B^2 - \alpha^2 \geq \phi_1^2$.

(iii) Finally we pass to charged dust distribution, obtained from (4.1) or (4.2) when α tends to zero

$$\mathrm{d}s^2 = \left(\mathrm{d}t + \Omega r^2\,\mathrm{d}\varphi\right)^2 - r^2\,\mathrm{d}\varphi^2 - \mathrm{e}^{Cr^2}(\mathrm{d}z^2 + \mathrm{d}r^2)\ , \tag{4.3}$$

with $2\Omega^2 = 2B^2 - 2C - \phi_1^2$, $2(B^2 - C) \geq \phi_1^2$ for $C \leq 0$, and $2(B^2 - 2C) \geq \phi_1^2$ for $0 \leq C \leq \frac{B^2}{2}$.

In each case (4.1)-(4.3) the ranges of C (or, equivalently, $\tilde{C}$) and ϕ_1 are indicated for which the energy conditions in table 1 are fulfilled and the signature is correct.

All of the above metrics (4.1)-(4.3) admit closed timelike curves: (4.1) as long as $4\Omega^2 > \alpha^2$, (4.2) and (4.3) for $\Omega \neq 0$ (see figure 1 and [11, 14] for details). Influence of the magnetic field to the chronology violation is positive unlike the scalar field influence, that is negative.

Metrics (4.1) or (4.3) suffer from singularities for $\tilde{C} < 0$ when r goes to infinity, whereas Riemann tensor invariants of (4.2) are well behaved (see [11]). Moreover (4.1) as well as (4.3) are radially bounded in the sense that the proper radial distance d of infinity $r \to \infty$ from the rotation axis is finite unless $\tilde{C} = 0$ (figure 2). For (4.3) the last assertion is obvious and for (4.1) it follows since d can be estimated as

$$d = \int_0^\infty |\mathrm{e}^\delta|\,\mathrm{d}r \leq \alpha^{-1}\int_0^\infty \frac{\mathrm{e}^{\tilde{C}x}}{\sqrt{x}}\,\mathrm{d}x < \infty\ , \quad \text{if } \tilde{C} < 0\ .$$

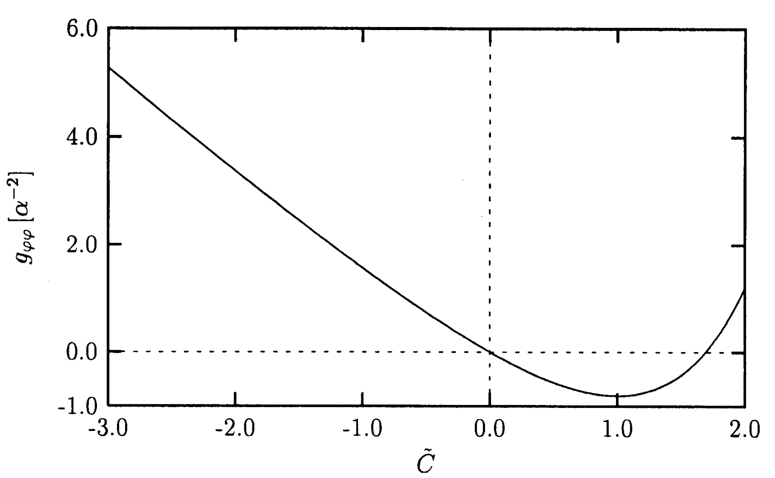

Figure 1: For metric (4.2) some of the circles t, z, r = const appear to be closed timelike curves e.g. if $r = r_n = (2n+1)\pi\alpha^{-1}$, n is integer. On figure the values of $g_{\varphi\varphi}(r_n)$ (in units α^{-2}) are plotted against $\tilde{C}$ in extreme case $2B^2 - \alpha^2 = \phi_1^2$. Only $\tilde{C} \leq 0$ range is physically permissible.

5 Concluding remarks

A few points should be mentioned, before concluding this paper.

As noted already before, metrics (3.5) and (3.10) contains arbitrary function as a consequence of the r-coordinate rescalling freedom. This arbitrarity can be locally removed by introducing a new coordinate R defined by implicit equation $\mathrm{d}R = \mathrm{e}^{\delta}\,\mathrm{d}r$.

To complete the picture, (3.1) also provides us with solutions (4.1)-(4.3) as its particular cases, namely with (4.1) subject to $2\Omega^2 = (1-\tilde{C})\alpha^2$, furthermore with (4.2) for $2\Omega^2 = -(1+\tilde{C})\alpha^2$, and with (4.3) when $\Omega^2 = -C$.
Dust solution (3.10) for $\sigma = 0$ contains (4.3) with $\Omega^2 = -C$.

In section 3 we have obtained the Wright solution alternative source. Thus by appropriate specialization of integration constants one can also obtain the Gödel solution with alternative source formed by the combination of scalar field and charged fluid. It was realized in [14] for z-dependent scalar field. Finally, we treat scalar field for the Gödel metric having (in contrast to [14]) non-trivial radial dependence. Let us take the Gödel limit $C = 0$ of (4.1) generated by (3.1) into account.[4] In this

[4]Metrics (4.2) and (3.10) do not generate Gödel-type solution besides the Minkowski spacetime.

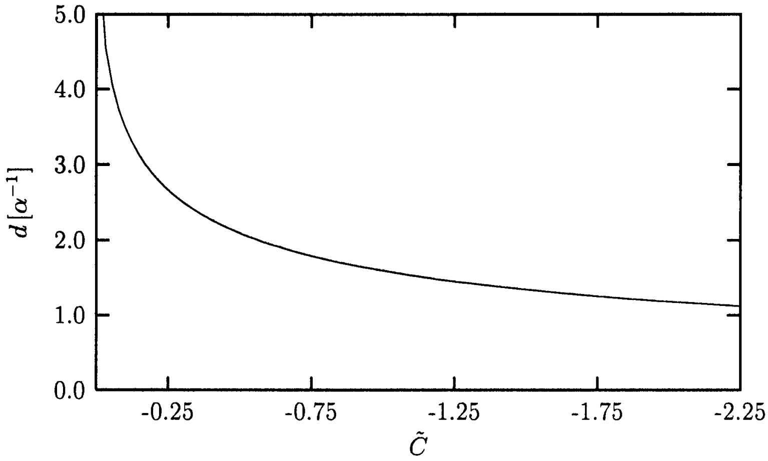

Figure 2: Smoothed dependence of proper radial distance d (in units α^{-1}) of infinity $r \to \infty$ from the rotation axis is plotted against $\tilde{C}$ for metric (4.1).

case the scalar field can be written explicitly. Then equation (2.8) results in

$$\phi = \phi_0 + \sqrt{2}Bz + \sqrt{2}F_{\varphi z} \ln \left| \text{th} \frac{\alpha r}{2} \right| \tag{5.1}$$

for the Gödel limit of (4.1). The equation (5.1) implies that ϕ diverges at $r = 0$ and in z-infinities, which is consequence of equations (3.3) and (3.4).

Note that the rest mass density (3.2) remains to be position dependent even after putting $C = 0$ due to $F_{\varphi z}$ term contribution. In spite of this the total energy density is uniform through the spacetime, as it must be, insuring the spacetime homogeneity.

Let us summarize briefly the main results of the paper. We focused on the stationary cylindrically symmetric spacetime for which the matter content is constituted by the charged perfect fluid and massless scalar field. We dealt with the Lorentz force-free case only, with the magnetic field (2.2). The fluid particles - carriers of the charge - rigidly rotate without acceleration. The solutions obtained are divided into two cases I and II. For the case I the radial magnetic field is compensated by radial part of scalar field gradient, for the case II not.

Case I consists of the Wright solution alternative source (3.1) and the generalization (3.5) of the Lorentz force-free solutions in [9, 11]. Case II describes charged dust solution (3.10) which is cylindrically symmetric spacetime but not regular at the origin.

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PROOFS

On spacetimes with 3-parameter isometry group in string-inspired theory of gravity[1]

P. Klepáč

Abstract. Cylindrically symmetric stationary spacetimes are examined in the framework of string-inspired generalized theory of gravity. In four dimensions this theory contains a dilatonic scalar field in addition to gravity. A charged perfect fluid representing fermionic matter is also considered. Explicit solution is given and a discussion of the geometrical properties of the solutions found is carried out.

Keywords. Cylindrical symmetry, exact solutions, first-order correction.
MS classification. 83C22, 83C15, 83E30.

1. Introduction

In this work some results on the stationary cylindrically symmetric spacetimes in string-inspired generalized theory of gravity are presented. The reason for studying this class of the spacetimes is two-fold. First, searching for stringy solutions is important in itself, see [1, 7]. Second, in classical relativity theory the cylindrically symmetric spacetimes are known to violate some of the chronology conditions, cf. [2].

Therefore it is natural to address the question of chronology violation in the stringy spacetimes. Barrow and Dąbrowski ([1]) had given string Gödel-type solutions and Kanti and Vayonakis ([3]) found the Gödel-type homogeneous metrics in the string-inspired charged gravity.

The paper is organized as follows. In Section 2 we consider an exact solution of a general relativistic system of a perfect fluid and a scalar field coupled to gravity. In Section 3 the results of the previous section are generalized on the α'-order corrections in the framework of the string-inspired theory, [7, 3]. Finally in Section 4 we treat some of the geometrical properties of the found solution.

[1] The paper is in final form and will not be published elsewhere.

2. Classical solution

Let spacetime $(\mathcal{M}, \mathbf{g})$ be connected 4-dimensional smooth orientable manifold endowed with lorentzian metric $\mathbf{g}$. We search for cylindrically symmetric stationary spacetimes. Then there exist local coordinate systems $(x^0, x^1, x^2, x^3) = (t, \varphi, z, r)$ adapted to Killing fields $\partial_t,\ \partial_\varphi,\ \partial_z$, where the hypersurfaces $\varphi = 0$ and $\varphi = 2\pi$ are to be identified and ∂_t is an everywhere non-vanishing timelike field.

Metric tensor field is expressed in the following way

$$\mathbf{g} = \eta_{\mu\nu}\Theta^\mu \otimes \Theta^\nu ,$$

where $(\eta_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$ is Minkowski matrix and Θ^μ are local coframe fields (greek indices run from 0 to 3) that form a (pseudo-)orthonormal basis in each cotangent space and are defined by

$$\begin{aligned} \Theta^0 &= \mathrm{d}t + f\,\mathrm{d}\varphi, & \Theta^1 &= l\,\mathrm{d}\varphi, \\ \Theta^2 &= \mathrm{e}^\gamma\,\mathrm{d}z, & \Theta^3 &= \mathrm{e}^\delta\,\mathrm{d}r, \end{aligned} \tag{1}$$

with $f,\ l,\ \gamma,\ \delta$ being functions of r only. Through the text the Einstein summation rule is used.

Our starting action is

$$S[\mathbf{g}, \phi, \mu] = \int_{\mathcal{M}} *R - \mathrm{d}\phi \wedge *\mathrm{d}\phi + 16\pi * \mu , \tag{2}$$

where R is the scalar curvature of the metric tensor $\mathbf{g}$, $\phi(r, z)$ and $\mu(r)$ are scalar fields on $\mathcal{M}$. From the physical point of view a coupled system of massless scalar field $\phi(z, r)$ and perfect fluid is considered. The fluid moves in its comoving system with velocity vector field $\mathbf{u} = \Theta^0$ and it is characterised by the pressure $p(r)$ and the energy density $\mu(r)$. (We use the same symbol for vector fields and their naturally corresponding 1-forms.)

The equations of motion for the metric field $\mathbf{g}$ are the Einstein field equations in basis (1) written as ([10])

$$-\tfrac{1}{2}\eta_{\alpha\beta\gamma} \wedge \Omega^{\beta\gamma} = 8\pi * i_\alpha \mathbf{T}, \tag{3}$$

where

$$\eta^{\alpha\beta\gamma} = *(\Theta^\alpha \wedge \Theta^\beta \wedge \Theta^\gamma),$$

Ω is curvature 2-form on $T\mathcal{M}$,

$$\Omega^\alpha_\beta = \tfrac{1}{2}R^\alpha_{\ \beta\gamma\delta}\Theta^\gamma \wedge \Theta^\delta ,$$

and $\mathbf{T}$ is the stress-energy tensor

$$8\pi\mathbf{T} = 8\pi\big[(\mu + p)\mathbf{u} \otimes \mathbf{u} - p\,\mathbf{g}\big] + \tfrac{1}{2}\big[\mathrm{d}\phi \otimes \mathrm{d}\phi - \tfrac{1}{2}\,\mathbf{g}(\mathrm{d}\phi, \mathrm{d}\phi)\,\mathbf{g}\big]. \tag{4}$$

The appropriate explicit form of the Einstein equations for the basis (1) is similar to the one in ([4]) so we simply refer the reader to the paper.

Bianchi identity $\mathrm{D} * i_\alpha \mathbf{T} = 0$ in our case implies $p = \text{const}$ because of the fluid particles geodesic motion, see below.

Altogether we obtain five independent equations for six unknown functions: f, l, γ, δ, μ and ϕ. Therefore the mathematical problem admits one degree of freedom corresponding to the r-coordinate rescaling possibility as can be seen directly from (1).

The scalar field equation of motion is the massless Klein–Gordon equation

$$* \mathrm{d} * \mathrm{d}\phi = 0 . \tag{5}$$

Because of the independence of r and z coordinates one can carry out the separation of variables in (5) to obtain

$$\phi = \phi_0 + \phi_1 z + \phi_2 \int l \, \mathrm{e}^{\delta-\gamma} \mathrm{d}r , \tag{6}$$

where ϕ_0, ϕ_1 and ϕ_2 are constants such that $\phi_1 \cdot \phi_2 = 0$.

For the function l^2 one gets a second-order non-linear equation that can be transformed into the following form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y' + x}{y}\right] = -\frac{k\phi_2^2}{y^2}, \qquad k = \text{const}. \tag{7}$$

We proceed further by dividing solutions of (7) into two groups according as ϕ_2 is zero, case (a), or is not, case (b).

Case (a): $\phi_2 = 0$

From (6) we can see that ϕ depends linearly on z alone. The solution of the Einstein equations can be settled in the form

$$\begin{aligned} \mathbf{g} &= \mathrm{d}t \otimes \mathrm{d}t + \frac{\Omega}{C}\gamma\,(\mathrm{d}t \otimes \mathrm{d}\varphi + \mathrm{d}\varphi \otimes \mathrm{d}t) - l^2\,\mathrm{d}\varphi \otimes \mathrm{d}\varphi \\ &\quad - \mathrm{e}^{2\gamma}\,\mathrm{d}z \otimes \mathrm{d}z - C^{-2} l^{-2}\,\mathrm{d}\mathrm{e}^{\gamma} \otimes \mathrm{d}\mathrm{e}^{\gamma} . \end{aligned} \tag{8}$$

The metric function l^2 is given by

$$l^2 = \frac{8\pi p}{C^2}\,\mathrm{e}^{2\gamma} - \frac{4\Omega^2 + \phi_1^2}{2C^2}\,\gamma + \nu , \tag{9}$$

with Ω, ν, C as integration constants and γ an arbitrary non-constant C^2-function.

Case (b): $\phi_2 \neq 0$

Now we have dust distribution ($p = 0$) that generalizes the van Stockum solution and the explicit form of the metric is omitted here for the reason given below. The scalar field is integrated to $e^{\phi} = (a\tilde{\gamma} + b)^{\phi_2/a}$, with $\tilde{\gamma}$ being an arbitrary function and a and b constants.

In this work we focus on the spacetimes with 3-parameter isometry group, especially on the stationary cylindrically symmetric ones. If one requires the found solutions to be cylindrically symmetric regular at the origin the axial symmetry condition and the elementary flatness condition have to be imposed ([5].) For Case (a) these conditions yield

$$8\pi p - \Omega^2 - \tfrac{1}{4}\phi_1^2 = C, \quad 8\pi p + C^2\nu = 0\,.$$

On the hand it turns out that Case (b) cannot be cylindrically symmetric.

For further investigation of the charged scalar field and allowed frequencies in Gödel-type background we refer the reader to ([6]), where is also considered the problem of field quantization using the Euclean approach to quantum field theory.

3. String-inspired theory

We consider a generalized theory of gravity which describes the coupling of a dilatonic scalar field to an electromagnetic field and gravity with the following α'-order corrected effective action in the Einstein frame ([3])

$$S_{\text{eff}}[\mathbf{g}, \mathbf{A}, \phi, \mu] = \int_{\mathcal{M}} [*R + 16\pi * \mu - \mathrm{d}\phi \wedge *\mathrm{d}\phi + 2\lambda\, e^{\phi}\, \mathbf{F} \wedge *\mathbf{F}]. \tag{10}$$

Here $\mathbf{F}$ is the electromagnetic field 2-form. Coupling constant λ is expressed like $\lambda = \alpha'/4g^2$, where α' is the inverse string tension (Regge slope) and g is essentially the string coupling constant. Physically, if the fluid particles are charge carriers, the charged perfect fluid is rough approximation of a fermionic matter in the theory.

We are interested in α'-order corrections to the classical solution (8). It means that in (10) we keep only zeroth and first order terms in α'.

The scalar field ϕ is written like $\phi = \phi^{(0)} + \alpha'\phi^{(1)}$, where $\phi^{(0)}$ is a classical zero-order solution. As Case (b) of Section 2 cannot represent cylindrically symmetric spacetime, we take $\phi^{(0)} = \phi_0 + \phi_1 z$. It should be mentioned that in some works (e.g., [7]) the additional term

$$S_{\text{GB}} = 8\pi^2\lambda \int_{\mathcal{M}} e^{\phi} e(\mathcal{M}),$$

enters action (10), where $e(\mathcal{M})$ is the Euler class of the manifold $\mathcal{M}$, in four-dimensions equal to

$$e(\mathcal{M}) = \frac{1}{8\pi^2}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2)\eta\,,$$

η is the volume element.

Next analysis will be simplified if one adopts the following reasonable conditions on the electromagnetic field behaviour. First, the Lorentz force $\mathcal{F}$ acting on the fluid particles, $\mathcal{F} \propto *(\mathbf{u} \wedge *\mathbf{F})$, vanishes (for the case with the non-vanishing Lorentz force but without the scalar field, see [4]). Second, only longitudinal magnetic field survives. In comoving system the magnetic field 1-form is given by $\mathbf{B} = *(\mathbf{u} \wedge \mathbf{F})$, which together with the previous point gives

$$*(\mathbf{u} \wedge \mathbf{F}) \wedge \Theta^2 = 0\,.$$

Before treating the equations of motion for the electromagnetic and gravitational fields we impose another supposition concerning the perfect fluid.

From the physical considerations it follows, because the fluid represents the fermionic matter, that it is more physically favourable if one assumes the presence of a continuous electric charge distribution throughout the spacetime with a charge current density 1-form $\mathbf{j}$. Nevertheless, to maintain the action (10) unaffected, the current density incorporation is permissible only in the case where the corresponding source term $\mathbf{j} \wedge *\mathbf{A}$ does not act as an extra gravitational field source.

Furthermore it is natural to postulate that the fluid particles are charge carriers. Thus the current density $\mathbf{j}$ is purely convectional, $\mathbf{j} = \rho\mathbf{u}$, with $\rho(r)$ being an invariant charge density.

With the above in mind the generalized Maxwell equations take the form

$$-*\mathrm{d}*(\mathrm{e}^{\phi^{(0)}}\mathbf{F}) = \frac{4\pi}{\lambda}\rho\,\Theta^0\,. \tag{11}$$

By virtue of the Einstein–Maxwell equations resulting from (10) (see below), the conditions above and (11) will be satisfied if

$$\mathbf{F} = B\exp(-\tfrac{1}{2}\phi_0 - \tfrac{1}{2}\phi_1 z - \gamma)\Theta^3 \wedge \Theta^1, \qquad B = \mathrm{const} \tag{12}$$

Note that

$$*\mathrm{d}\mathbf{F} = \tfrac{1}{2}B\phi_1\exp(-\phi^{(0)}/2 - 2\gamma)\,\Theta^0\,, \tag{13}$$

which signals that we deal with a magnetic monopoles current. As a matter of fact it is neccessary to introduce a magnetic charges current density 1-form $\mathbf{j}_m$. In principle there are possibilities to retain (13) physically admissible. Either we can expect that going to a non-Abelian gauge fields will smooth out this solution, or, in the case of Abelian gauge fields, it is possible to introduce $\mathbf{j}_m$ explicitly in the action (10), but one has to break the general covariance to do this ([9]).

The metric field equations of motion following from the superstring effective action (10) are the Einstein equations (3) enriched by the electromagnetic field contribution ([10])

$$8\pi\mathbf{T}_{\mathrm{elmag}} = \lambda\frac{\mathrm{e}^{\phi^{(0)}}}{8\pi}\,\Theta^\alpha \otimes *(\mathbf{F} \wedge i_\alpha *\mathbf{F} - i_\alpha\mathbf{F} \wedge *\mathbf{F})$$

to the stress-energy tensor field (4). Again we refer the reader to ([4]) for details.

One finds that one has six independent equations for totality of seven unknowns: four metric functions $f,\ l,\ \gamma,\ \delta$ and three physical quantities $\phi,\ \mu,\ \rho$. Again, we have one degree of freedom due to the r-coordinate rescaling possibility.

Bianchi identity in our case, provided that the scalar field equation of motion is fulfilled, is

$$\mathbf{u} \wedge *(\mu\, \mathrm{d}\mathbf{u} + \mathrm{d}(p\,\mathbf{u}) + \lambda\, \rho\, \mathrm{e}^{\phi^{(0)}}\, \mathbf{F}) = 0\,. \tag{14}$$

Inserting (1) and (12) into (14) one finds that the pressure p is constant.

The mathematical structure of the Einstein–Maxwell equations is much the same as classical theory, Section 2 (see also [4]). As a consequence, if one requires solution to be cylindrically symmetric, the scalar field becomes equal to

$$\phi = \phi_0 + \tilde{\phi}_1 z\,,$$

where $\tilde{\phi}_1$ is constant.

Resulting metric is given by

$$\begin{aligned} \mathbf{g} &= \mathrm{d}t \otimes \mathrm{d}t + \frac{\Omega}{C}\gamma\,(\mathrm{d}t \otimes \mathrm{d}\varphi + \mathrm{d}\varphi \otimes \mathrm{d}t) - l^2\, \mathrm{d}\varphi \otimes \mathrm{d}\varphi \\ &\quad - \mathrm{e}^{2\gamma}\, \mathrm{d}z \otimes \mathrm{d}z - C^{-2} l^{-2}\, \mathrm{d}\mathrm{e}^{\gamma} \otimes \mathrm{d}\mathrm{e}^{\gamma}\,, \\ l^2 &= \frac{8\pi p}{C^2}\,\mathrm{e}^{2\gamma} - \frac{4\Omega^2 + \tilde{\phi}_1^2 - 4\lambda B^2}{2C^2}\,\gamma + \nu. \end{aligned} \tag{15}$$

Found solution (15) is very similar to the zero-order solution (8). The only difference appears in l^2 function (9), namely the coefficient of the linear term in γ gets shifted due to the magnetic field presence.

The formula for the energy density

$$\mu = \frac{1}{4\pi}(2\Omega^2 - \lambda B^2)\mathrm{e}^{-2\gamma} - 3p$$

has very transparent physical interpretation. A "specific" mass density $\mu + 3p$ must be added to the magnetic energy density to balance the rotation.

As an important example of (15) we choose the integration constants like

$$\begin{aligned} &16\pi p = -C^2 \nu = m^2, \qquad \gamma = \frac{2C}{m^2}\,\mathrm{sh}\left(\frac{mr}{2}\right), \\ &4\Omega^2 = 4\lambda B^2 - \tilde{\phi}_1^2 + 2m^2(1 - C), \end{aligned} \tag{16}$$

where m is constant. With this choice the energy density becomes equal to

$$8\pi\mu = \left(2\lambda B^2 - \tilde{\phi}_1^2 + 2m^2[1 - C]\right)\mathrm{e}^{-2\gamma} - \tfrac{3}{2}m^2. \tag{17}$$

In (16) and (17) the substitution $Cm^{-2} \to C$ is assumed. Since the pressure is constant it follows that our spacetime can be reinterpreted as the charged dust solution (with the vanishing pressure) and the non-zero cosmological constant, [4].

Note that if one wants at this stage to eliminate the perfect fluid contribution to the action (10) and consider only coupling of the electromagnetic and the dilatonic fields to gravity in a spacetime with the non-zero cosmological constant the following equality must hold

$$\mu + p = 0. \tag{18}$$

In this case the equations (17) and (18) imply $C = 0$ along with $\tilde{\phi}_1^2 = 2\lambda B^2 + m^2$. The metric tensor is found to be

$$\begin{aligned} \mathbf{g} = &\left(\mathrm{d}t + \frac{4\Omega}{m^2}\,\mathrm{sh}\left(\frac{mr}{2}\right)\mathrm{d}\varphi\right) \otimes \left(\mathrm{d}t + \frac{4\Omega}{m^2}\,\mathrm{sh}^2\left(\frac{mr}{2}\right)\mathrm{d}\varphi\right) \\ &- \frac{1}{m^2}\,\mathrm{sh}^2(mr)\,\mathrm{d}\varphi \otimes \mathrm{d}\varphi - \mathrm{d}z \otimes \mathrm{d}z - \mathrm{d}r \otimes \mathrm{d}r, \end{aligned} \tag{19}$$

with Ω subject to $4\Omega^2 = 2\lambda B^2 + m^2$. The dilaton is approximately given by

$$\phi = \phi_0 + \left(\phi_1 + \frac{\lambda B^2}{m}\right) z\,. \tag{20}$$

The solution (19) and (20) manifestly describes the Gödel-type spacetime ([8]) and was found (in [3]) by another way when studying a homogeneous Gödel-type solutions. Note that in zero-order regime $\lambda \to 0$ one has $\phi_1^2 = m^2$ and $4\Omega^2 = m^2$, the latter equality immediately implying

$$g_{\varphi\varphi} = \partial_\varphi \cdot \partial_\varphi \le 0.$$

Thus there are no closed time-like curves in the spacetime, [2].

On the other hand in α'-order framework $g_{\varphi\varphi}$ becomes positive for sufficiently large r (because λ is positive). In this way the first-order correction causes the chronology violation.

4. Geometrical properties

In each tangent space $T_p\mathcal{M}$ the projection tensor onto 3-dimensional subspaces W_p, orthogonal to $\mathbf{u}$, is given by

$$\mathbf{h} = \mathbf{g} - \mathbf{u} \otimes \mathbf{u}\,.$$

The tensor field $-\mathbf{h}$ serves a positive definite metric on W_p.

For a given spacetime to be static there must exist hypersurface-orthogonal time-like Killing vector field, [5]. In our case the vorticity 1-form ω equals

$$\omega = \tfrac{1}{2} * (\mathbf{u} \wedge \mathbf{du}) = \Omega\,\mathrm{d}z.$$

In this way the metric (15) is static if and only if $\Omega = 0$. Geometrically, a collection $W = \bigcup_p W_p$ is not involutive.

The fluid particles acceleration $\dot{\mathbf{u}}$ is given by

$$\dot{\mathbf{u}} = - * (\mathbf{u} \wedge *\mathbf{du}),$$

and vanishes for (1) showing that $\mathbf{u}$ parallelly propagates along itself and fluid particles move geodesically. Namely for this reason, and because we have restricted our attention to the Lorentz force-free case, the pressure has to be constant.

The rate-of-strain tensor field is actually the extrinsic curvature $\mathbf{K}$ defined by

$$\mathbf{K} = \tfrac{1}{2} \underset{\mathbf{u}}{\pounds} \mathbf{h}.$$

Direct computation shows that $\mathbf{K}$ vanishes identically for (15) which from the physical viewpoint means that the fluid rotates as a rigid body.

The last remark concerns the algebraic classification of the Weyl tensor field $\mathbf{C}$. It turns out that for (15) there generally exist just four distinct null vectors $\mathbf{k}$ (modulo multiplying $\mathbf{k} \to a\mathbf{k}$, a is constant) satisfying

$$k^\beta k^\gamma k_\lambda C_{\alpha\beta\gamma\delta} k_\sigma (\Theta^\lambda \wedge \Theta^\alpha) \otimes (\Theta^\delta \wedge \Theta^\sigma) = 0.$$

Thus our spacetime (15) belongs to the type I according to the Petrov classification, cf. [5].

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A cylindrically symmetric solution in Einstein-Maxwell-dilaton gravity[4]

P. Klepáč [1] [2] and J. Horský [1] [3]

We consider the existence of Einstein-Maxwell-dilaton plus fluid system for the case of stationary cylindrically symmetric spacetimes. An exact inhomogeneous ε-order solution is found, where the parameter ε parametrizes the non-minimally coupled electromagnetic field. Some its physical attributes are investigated and a connection with already known Gödel-type solution is given. It is shown that the found solution also survives in the string-inspired charged gravity framework. We find that a magnetic field has positive influence on the chronology violation unlike the dilaton influence.

KEY WORDS: exact solutions, charged perfect fluid, scalar field

[4]Suggested running head: Cylindrically symmetric solution in EMD gravity

[1]Institute of Theoretical Physics and Astrophysics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic

[2]E-mail: klepac@physics.muni.cz

[3]E-mail: horsky@physics.muni.cz

1 Introduction

Einstein's theory of relativity is, in general, an excellent approximation of gravitational phenomena which appear at low energies. Nevertheless, if one goes to energies at the Planck scale, then one is faced with the neccessity of introducing quantum corrections. Nowadays, superstring theory is believed to unify successfully all of the known fundamental interactions observed in nature. Moreover the original Einstein's theory naturally emerges if one ignores all higher-order stringy corrections.

In the last decade the string cosmology has become an attractive subject of interest. As one goes to the low energies, string cosmology is actually the classical cosmology with general relativity enriched by addition of massless scalar fields. An incorporation of these fields is perhaps promissible way how to resolve long standing problems in cosmology.

A number of solutions to the so called Einstein-Maxwell-dilaton gravity, i.e. relativistic gravity theory containing non-minimally coupled dilaton and electromagnetic fields, has been derived by various techniques [1]. Barrow and Dąbrowski [2] obtained stringy Gödel-type solution without closed timelike curves (CTC's) by considering the one-loop corrected superstring effective action. Kanti and Vayonakis [3] have extended analysis of [2] on case with an electromagnetic field, and their found that the chronology is violated. Also results of others authors show that presence of the electromagnetic field may cause the chronology violation (especially purely magnetic filed parallel to the rotation axis) while a scalar field may again restore the chronology [4, 5].

In this work some results on the stationary cylindrically symmetric spacetimes in Einstein-Maxwell-dilaton (EMD) theory of gravity are presented. The reason for studying this class of the spacetimes is twofold. First, searching for EMD solutions is important in itself. Second, in the classical relativity theory the cylindrically symmetric spacetimes are known to violate some of the chronology conditions [6]. Therefore it is natural to address the question of chronology violation in the EMD spacetimes. The paper extends the results of [2, 3] to the inhomogeneous case where, in general, only three isometries are present.

The paper is organized as follows. After some preliminaries in section 2 we derive in section 3 the exact solution for the lowest order in parameter ε, which parametrizes the electromagnetic field. In section 4 there are the results of the previous one generalized on the ε-order corrections in the framework of the EMD theory. The case with more scalar fields is studied in section 5 provided that in zero-order they depend solely on the longitudinal direction. In section 6 it is shown that the found solution in fact still applies even if string-inspired charged gravity is taken under consideration. Finally section 7 briefly summarizes the basic properties of this solution.

2 Preliminaries

We search for cylindrically symmetric stationary spacetimes. Then there exist local coordinate systems $(x^0, x^1, x^2, x^3) = (t, \varphi, z, r)$ adapted to Killing fields ∂_t, ∂_φ, ∂_z, where the hypersurfaces $\varphi = 0$ and $\varphi = 2\pi$ are to be identified and ∂_t is everywhere a nonvanishing timelike field.

Furthermore we choose a local coframe fields $\Theta^{\hat{\mu}}$ defined by (Greek indices run from 0 to 3)

$$\begin{aligned} \Theta^{\hat{0}} &= e^{\alpha}(dt + f\, d\varphi) \ , & \Theta^{\hat{1}} &= l\, d\varphi \ , \\ \Theta^{\hat{2}} &= dz \ , & \Theta^{\hat{3}} &= e^{\delta}\, dr \ , \end{aligned} \tag{2.1}$$

with f, l, α, δ being functions of r only.

Let the metric tensor field be in the basis (2.1) written as $g = \eta_{\mu\nu}\Theta^{\hat{\mu}} \otimes \Theta^{\hat{\nu}}$, where $(\eta_{\mu\nu}) = \mathrm{diag}(\ 1,\ -1,\ -1,\ -1)$ is the Minkowski matrix.

The defining equations (2.1) show that in this paper the g_{zz} metric field component is constant, in contrast to [5, 7], where the case with constant g_{tt} component was studied.

Let the spacetime is filled with the charged perfect fluid and massless scalar field ϕ. The perfect fluid is characterized by its pressure p and energy density μ, from symmetry considerations both these quantities depending only on the radial coordinate. On the other hand ϕ may depend also on the longitudinal direction z. The electromagnetic field is non-minimally coupled to ϕ with parameter ε. Here we would like to point out that the spacetime with basis fields (2.1) is referred to cylindrically symmetric althought the dilaton generally depends also on the longitudinal direction.

The ε-order EMD action that we will deal with is given by

$$S_{\text{eff}}[g, A, \phi, \mu] = \int_{\mathcal{M}} \left[*R + 16\pi * \mu - d\phi \wedge * d\phi + 2\varepsilon\, e^{\phi}\ F \wedge *F \right] \ , \tag{2.2}$$

where ε is a real parameter, R is the Ricci scalar of the metric tensor, F is the electromagnetic field 2-form. The scalar field ϕ will be henceforth called dilaton.

We claim to obtain a EMD solution which is of the first order in the parameter ε. Of course at the first place it means one should have a zero-order solution in ε that solves equations of motion for the action (2.2) if we let ε to be zero, or in other words, a purely classical solution of the Einstein equations coupled with a scalar field and perfect fluid. Then the ε-order solution we are looking for is naturally viewed as electromagnetic first order correction of the classical solution, and it has to coincide with the latter when ε goes to zero.

As for the dilaton ϕ, it may be written as the sum of zero-order solution $\phi^{(0)}$ and a ε-order correction like

$$\phi = \phi^{(0)} + \varepsilon\phi^{(1)} \ . \tag{2.3}$$

If only terms linear in ε are considered, one is forced to keep terms of ε-order in the corresponding equations of motion. Particularly, the coupling fuction standing at $F \wedge *F$, being already of $\mathcal{O}(\varepsilon)$ becomes in our approximation equal to $2\varepsilon \mathrm{e}^{\phi^{(0)}}$.

An usual progress is to introduce a fluid comoving system, in which the fluid particles motion is uniquely determined by the velocity (co)vector field u, $u = \Theta^{\hat{0}}$. Let us very briefly mention basic properties of the geometry of fluid particles worldlines congruences. An acceleration 1-form $\dot{u}$ is given as $\dot{u} = -\,\mathrm{d}\alpha$. Since the problem is stationary, both expansion and shear tensor are vanishing. A vorticity covector is given by

$$\omega = \frac{1}{2} * (u \wedge \mathrm{d}u) = \frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}r} l^{-1} \mathrm{e}^{\alpha-\delta}\,\mathrm{d}z \ . \tag{2.4}$$

The last two statements show that the fermionic fluid rotates as a rigid body.

3 Zeroth-order solution

In this paragraph we derive a solution of zero-order in ε. In this case the action (2.2) becomes Einstein-dilaton plus fluid system. The Einstein field equations written in the tetrad representation (2.1) then read

$$-\frac{1}{2}\,\eta_{\alpha\beta\gamma} \wedge \Omega^{\hat{\beta}\hat{\gamma}} = 8\pi * i_{\hat{\alpha}} T \ , \tag{3.1}$$

where $i_{\hat{\alpha}} \equiv i_{\mathbf{e}_{\hat{\alpha}}}$ is the interior product ($\mathbf{e}_{\hat{\alpha}}$ is dual basis to (2.1), $\Theta^{\hat{\alpha}}(\mathbf{e}_{\hat{\beta}}) = \delta^{\alpha}_{\beta}$), Ω is curvature 2-form on $T\mathcal{M}$, $\Omega^{\hat{\alpha}}_{\hat{\beta}} = \frac{1}{2} R^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}\hat{\delta}} \Theta^{\hat{\gamma}} \wedge \Theta^{\hat{\delta}}$, and 1-form $\eta^{\alpha\beta\gamma}$ is defined by [8]

$$\eta^{\alpha\beta\gamma} = *(\Theta^{\hat{\alpha}} \wedge \Theta^{\hat{\beta}} \wedge \Theta^{\hat{\gamma}}) \ .$$

Finally T is the total stress-energy tensor of the perfect fluid and the masseless scalar field,

$$\begin{aligned} 8\pi T &= 8\pi\left[(\mu + p) u \otimes u - p\, g\right] \\ &+ \tfrac{1}{2}\left[\mathrm{d}\phi \otimes \mathrm{d}\phi - \tfrac{1}{2}\, g(\mathrm{d}\phi, \mathrm{d}\phi)\, g\right] \ . \end{aligned} \tag{3.2}$$

Explicit form of the Einstein equations takes the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{e}^{2\alpha}}{M}\frac{\mathrm{d}f}{\mathrm{d}r}\right] = 0 \ , \tag{3.3a}$$

$$\frac{2}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{M}\frac{\mathrm{d}\alpha}{\mathrm{d}r}\right] = -\frac{1}{M^2}\left(\frac{\partial\phi}{\partial r}\right)^2 , \tag{3.3b}$$

$$\frac{1}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{e}^{-2\alpha}}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left(l^2\mathrm{e}^{2\alpha}\right)\right] = 32\pi p\,\mathrm{e}^{2\alpha} \ , \tag{3.3c}$$

$$l^2\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial z} = 0 \ , \tag{3.3d}$$

$$\frac{\mathrm{e}^{-2\alpha}}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{M}\frac{\mathrm{d}l^2}{\mathrm{d}r}\right] = 16\pi(p - \mu) + \frac{1}{M^2}\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2 , \tag{3.3e}$$

$$2\mathrm{e}^{-2\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}r}\frac{\mathrm{d}\,l^2}{\mathrm{d}r} = 32\pi M^2 p - \left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 + l^2\left(\frac{\partial\phi}{\partial r}\right)^2 \mathrm{e}^{-2\alpha} - \left(\frac{\partial\phi}{\partial z}\right)^2 M^2 \;, \tag{3.3f}$$

Let us introduce the functions m and M by the formulae

$$M = l\mathrm{e}^{\delta-\alpha}\;, \qquad m = \int M\,\mathrm{d}r\;.$$

The scalar field equation of motion is the massless Klein-Gordon equation

$$*\;\mathrm{d}*\mathrm{d}\;\phi = \frac{1}{M}\frac{\partial}{\partial r}\left(\frac{l^2}{M}\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial z^2} = 0\,. \tag{3.4}$$

The Bianchi identity, provided that the dilatonic equation of motion (3.4) is satisfied, becomes

$$u \wedge *\left[\mu\,\mathrm{d}u + \mathrm{d}(pu)\right] = 0\;. \tag{3.5}$$

Because of the independence of r and z coordinates in (3.3) one can carry out the separation of variables in (3.4) to obtain ϕ in terms of the metric functions,

$$\phi = \phi_0 + \phi_1 z + \phi_2 \int \frac{M}{l^2}\,\mathrm{d}r\,, \quad \phi_1\phi_2 = 0\;, \tag{3.6}$$

with constants ϕ_0, ϕ_1 and ϕ_2.

Thus one has reduced the problem to solving five equations (3.3a)-(3.3f) minus (3.3d) for six unknowns: α, f, l, δ and physical quantities of pressure p and mass (energy) density μ.

Inserting of (3.3a), (3.3b) and (3.3c) into (3.3f) yields system of two second-order equations for α and l^2 that reads

$$2l^4\frac{\mathrm{d}^2\alpha}{\mathrm{d}m^2} = -{\phi_2}^2\;, \tag{3.7a}$$

$$\frac{\mathrm{d}^2 l^2}{\mathrm{d}m^2} = {\phi_1}^2\mathrm{e}^{2\alpha} + 4\Omega^2\mathrm{e}^{-2\alpha}\;, \tag{3.7b}$$

The authors have been able to find a solution to (3.7) if $\phi_2 = 0$, which from (2.3) and (3.6) immediately implies

$$\phi^{(0)} = \phi_0 + \phi_1 z\;. \tag{3.8}$$

This especially simple linear dependence of the dilaton is common in papers [2, 3, 4]. It also naturally emerges once one admits the dilatonic dependence only on the coordinate along the rotation axis. Since the dilaton blows up at the z-infinities, they can be condidered as additional sources of scalar charge.

The solution of the Einstein equations can be written in the form

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{e}^{2\alpha}\left(\mathrm{d}t + f\,\mathrm{d}\varphi\right)^2 - l^2\,\mathrm{d}\varphi^2 \\ &- \mathrm{d}z^2 - C^{-2}l^{-2}\left(\mathrm{d}\mathrm{e}^{\alpha}\right)^2\;, \end{aligned} \tag{3.9}$$

the metric functions f and l^2 being given by

$$f = -\frac{\Omega}{C}e^{-2\alpha} + F \ , \tag{3.10a}$$

$$C^2 l^2 = \Omega^2 e^{-2\alpha} + \frac{1}{4}\phi_1^2 e^{2\alpha} + D\alpha + E \ , \tag{3.10b}$$

with Ω, C, D, E, F integration constants. The physical quantities, the energy density and the pressure, are found to be

$$\begin{aligned} 16\pi\mu &= De^{-2\alpha} - \phi_1^2 \ , \\ 16\pi p &= De^{-2\alpha} + \phi_1^2 \ . \end{aligned} \tag{3.11}$$

The formulae (3.10) and (3.11) are expressed in terms of an arbitrary non-constant C^2 function α that reflects the radial coordinate rescaling possibility.

4 First-order solution

An electromagnetic field is represented by a 2-form F in the action (2.2). The electromagnetic field, being already of ε-order, is non-minimally coupled to gravity with the firm (exponential) dependence on the longitudinal direction. Our next task is to take a suitable *Ansatz* for the electromagnetic field and then solve the equations of motion.

Let charge be distributed with a current density $j(r,z)$ through a spacetime. Note we have not included the source term $A \wedge *j$, where A is a vector potential, into the action (2.2) because of technical simplicity. This is possible if and only if $A \wedge *j$ is an exact form and can be transformed away.

Of great physical importance, in particular on the field of rotating spacetimes we deal with, is the case when the Lorentz force, in the comoving system proportional to $*(u \wedge *F)$, acting on the fluid particles, vanishes. In the fluid rest frame it means that only a magnetic field survives. The form of the metric field equations of motion, namely the φz and φr components, leads us to exclude the spacetime with electric currents parallel to the axis of rotation, in which the angular part of the magnetic field vanishes identically. The electromagnetic field 2-form is then given by

$$F = \mathrm{B}_{\hat{r}}\ \Theta^{\hat{1}} \wedge \Theta^{\hat{2}} + \mathrm{B}_{\hat{z}}\ \Theta^{\hat{3}} \wedge \Theta^{\hat{1}} \ . \tag{4.1}$$

The presence of the radial magnetic field may seem to be artificial because it causes a strange phenomena - an occurence of magnetic charges (monopoles). In fact, this is the case. But the form of zr-component of the Einstein equations, namely the equation (4.2d), enforces the existence of the radially pointing magnetic field in order for the dilaton to be also radially dependent. Otherwise it would simply be given by (3.8).

The metric field equations of motion following from the action (2.2) are the Einstein equations (3.1) with the stress-energy tensor (3.2) enriched by the electromagnetic field contribution [8], where electromagnetic field is non-minimally coupled to gravity,

$$T_{\mathrm{elmag}} = \varepsilon \frac{\mathrm{e}^{\phi^{(0)}}}{8\pi}\ \Theta^{\hat{\alpha}} \otimes *(F \wedge i_{\hat{\alpha}} * F - i_{\hat{\alpha}}\ F \wedge *F)\ .$$

The appropriate Einstein-Maxwell system reads

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{e}^{2\alpha}}{M}\frac{\mathrm{d}f}{\mathrm{d}r}\right] = 0\ , \tag{4.2a}$$

$$\frac{2}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{M}\frac{\mathrm{d}\alpha}{\mathrm{d}r}\right] + \frac{1}{M^2}\left(\frac{\partial\phi}{\partial r}\right)^2 = \frac{4\varepsilon}{l^4}\mathrm{e}^{\phi+2\alpha}F_{\varphi z}^2\ , \tag{4.2b}$$

$$\frac{1}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{e}^{-2\alpha}}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left(l^2\mathrm{e}^{2\alpha}\right)\right] = 32\pi p\,\mathrm{e}^{2\alpha}\ , \tag{4.2c}$$

$$l^2\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial z} = 4\varepsilon\mathrm{e}^{\phi}F_{r\varphi}F_{\varphi z}\ , \tag{4.2d}$$

$$\frac{\mathrm{e}^{-2\alpha}}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{M}\frac{\mathrm{d}l^2}{\mathrm{d}r}\right] = 16\pi(p-\mu) + \frac{1}{M^2}\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2 - 4\varepsilon\mathrm{e}^{\phi}\frac{F_{\varphi z}^2}{l^2}, \tag{4.2e}$$

$$\begin{aligned} 2\mathrm{e}^{-2\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}r}\frac{\mathrm{d}\,l^2}{\mathrm{d}r} &= 32\pi M^2 p - \left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 + 4\varepsilon\mathrm{e}^{\phi-2\alpha}F_{r\varphi}^2 - 4\varepsilon\mathrm{e}^{\phi}\frac{M^2}{l^2}F_{\varphi z}^2 \\ &+ l^2\left(\frac{\partial\phi}{\partial r}\right)^2\mathrm{e}^{-2\alpha} - \left(\frac{\partial\phi}{\partial z}\right)^2 M^2\ , \end{aligned} \tag{4.2f}$$

and it is to be completed by the massless Klein-Gordon equation (3.6), which does not undergo any changes, and furthermore by the modified Maxwell equations. In (4.2) as well as in the remainder of this section e^{ϕ} stands for $\mathrm{e}^{\phi^{(0)}}$.

Variation of the action (2.2) with respect to a vector potential A yields the generalized Maxwell equations

$$- * \mathrm{d} * (\mathrm{e}^{\phi^{(0)}}\ F) = \frac{4\pi}{\varepsilon}\ j\ , \tag{4.3}$$

or in the explicit form

$$\begin{aligned} &l^2\mathrm{e}^{\phi-2\alpha}\frac{\partial}{\partial r}\left[(f\delta_0^{\mu} - \delta_1^{\mu})\frac{F_{r\varphi}}{M}\right] \\ &+M(f\delta_0^{\mu} - \delta_1^{\mu})\frac{\partial}{\partial z}(\mathrm{e}^{\phi}F_{z\varphi}) = -\frac{4\pi}{\varepsilon}Ml^2 j^{\mu}\ . \end{aligned} \tag{4.4}$$

The same procedure as in the zero-order case gives the following system for functions l^2 and α

$$2l^4\frac{\mathrm{d}^2\alpha}{\mathrm{d}m^2} = 4\varepsilon\mathrm{e}^{\phi+2\alpha}F_{\varphi z}^2 - \tilde{\phi}_2^2\ , \tag{4.5a}$$

$$\frac{\mathrm{d}^2 l^2}{\mathrm{d}m^2} = \tilde{\phi}_1^2\mathrm{e}^{2\alpha} + 4\Omega^2\mathrm{e}^{-2\alpha} - 4\varepsilon\mathrm{e}^{\phi}\frac{F_{r\varphi}^2}{M^2} \tag{4.5b}$$

with new constants $\tilde{\phi}_1$ and $\tilde{\phi}_2$ which already include the ε-order correction.

Because the dilaton is written as (2.3) and $\phi^{(0)}$ does not depend on r, the term $(\frac{\partial\phi}{\partial r})^2$ (and also $\tilde{\phi}_2^2$) is already of $\mathcal{O}(\varepsilon^2)$ and should be neglected. Putting together equations (4.2d), (4.5a) and (4.4) we have arrived at the following conditions for the electromagnetic field

$$\begin{aligned} *(u \wedge F) \wedge \Theta^{\hat{2}} = 0 \ , \\ u \wedge *F = 0 \ . \end{aligned} \tag{4.6}$$

The equations (4.2) and (4.6) are solved by a purely longitudinal magnetic field, parallel to the rotation axis

$$\mathrm{B}_{\hat{z}} = B \mathrm{e}^{-\frac{1}{2}\phi^{(0)} - \alpha} \Theta^{\hat{3}} \wedge \Theta^{\hat{1}} \ , \qquad B = \mathrm{const} \tag{4.7}$$

while the radially pointing magnetic field vanishes, i.e. $\mathrm{B}_{\hat{r}} = 0$ in (4.1). As a matter of fact one has quite transparent physical interpretation of the found result. Since according to (4.7) and (4.3) it must hold $j \wedge \Theta^{\hat{0}} = 0$, we conclude that the fluid particles are the charge carriers, i.e. the current density is purely convectional, $j = \rho u$. The charge density ρ is determined by the formula

$$4\pi\rho = -\varepsilon \frac{B}{M} \frac{\mathrm{d}f}{\mathrm{d}r} \mathrm{e}^{\frac{1}{2}\phi^{(0)} - \alpha} \ . \tag{4.8}$$

But there is a price we must pay for the simplification. Note that the exterior derivative of (4.7) does not vanish which means that we deal with a current of the magnetic monopoles and one has to introduce a magnetic charges current density 1-form j_m by

$$\frac{4\pi}{\varepsilon} \, j_m = *\,\mathrm{d}F = -\frac{1}{2} B \tilde{\phi}_1 \mathrm{e}^{-\frac{1}{2}\phi^{(0)} - \alpha} \Theta^{\hat{0}} \ . \tag{4.9}$$

Equations (4.8) and (4.9) show us that the source term $A \wedge *j$ is identically vanishing.

Essentially there are two possibilities to keep this situation physically acceptable. Either we can expect that going to a non-abelian gauge fields will smooth out this solution, or, in the case of abelian gauge fields, it is possible to introduce j_m explicitly in the action (2.2), but one has to break the general covariance to do this [9].

Now we can straightforwardly solve the Einstein equations. From the same reason as in the zero-order case one has one degree of freedom corresponding to the radial coordinate rescalling possibility. One finds that one has seven independent equations for exactly eight unknowns: four metric functions f, l, α, δ and four physical quantities p, ϕ, μ, ρ.

The dilaton according to the (3.6) becomes equal to

$$\phi = \phi_0 + \tilde{\phi}_1 z \ .$$

The Bianchi identity in our case, provided that the scalar field equation of motion (3.6) is fulfilled, is

$$u \wedge *(\mu \ \mathrm{d}u + \mathrm{d}(p \ u) + \varepsilon \ \rho \ \mathrm{e}^{\phi^{(0)}} \ F) = 0 \ . \tag{4.10}$$

After all one obtains the result (3.9) with the following functions f and l

$$\begin{aligned} f &= -\frac{\Omega}{C}\mathrm{e}^{-2\alpha} + F \ , \\ C^2 l^2 &= \Omega^2 \mathrm{e}^{-2\alpha} + \frac{1}{4}\tilde{\phi}_1^2 \mathrm{e}^{2\alpha} - 2\varepsilon B^2 \alpha^2 + D\alpha + E. \end{aligned} \tag{4.11}$$

For the energy density, the pressure and the charge density (4.8) one has

$$\begin{aligned} 16\pi\mu &= \left[D + 2\varepsilon B^2 (1 - 2\alpha)\right] \mathrm{e}^{-2\alpha} - \tilde{\phi}_1^2 \ , \\ 16\pi p &= \left[D - 2\varepsilon B^2 (1 + 2\alpha)\right] \mathrm{e}^{-2\alpha} + \tilde{\phi}_1^2 \ , \\ 2\pi\rho &= -\varepsilon \Omega B \mathrm{e}^{\frac{1}{2}\phi^{(0)} - 3\alpha} \ . \end{aligned} \tag{4.12}$$

The mathematical structure of the solution (4.11) and (4.12) of the Einstein-Maxwell equations (4.2) is much the same as the zeroth-order one (3.10) and (3.11). The differences appear in the presence of the term quadratic in α in the function l^2 in (4.11) and the linear terms in α in (4.12), which occurs due to the existence of the magnetic field. The function f remains unchanged.

5 Case with more scalar fields

It is straightforward to generalize the zeroth order solution (3.10) and the first order solution (4.11) in the case when more scalar fields are present. The motivation comes from string theory, where it is known that the effective description at low energies may contain not only the dilaton, but also others tensor fields, depending on how the compactification was carried out [10]. Among them is most important an axionic tensor field that can be, just in four dimensions, represented by an extra massless scalar field. Also some additional massless scalar fields, called modulus fields, may be present [3].

We shall consider N massless scalar fields ϕ_i, $i = 1, 2, ..N$, and N non-minimally coupled massless scalar fields ψ_i. The total action (2.2) can be rewritten as

$$\begin{aligned} S_{\text{eff}} = \int_{\mathcal{M}} &\left[*R + 16\pi * \mu + 2\varepsilon \ \mathrm{e}^{\phi} \ F \wedge *F \right. \\ &\left. - \sum_i (\, \mathrm{d}\phi_i \wedge * \mathrm{d}\phi_i + \mathrm{e}^{-2\phi_i} \, \mathrm{d}\psi_i \wedge * \mathrm{d}\psi_i) \right] \ . \end{aligned} \tag{5.1}$$

Before proceeding further let us mention that each scalar field ϕ_i or ψ_i can be written a similar way to equation (2.3).

5.1 Zeroth order in ε

The Klein-Gordon equation (3.4) for each of the scalar fields ϕ_i still holds, while the equations of motion for the scalar fields ψ_i are given by

$$* \, \mathrm{d}(\mathrm{e}^{-2\phi_i} * \, \mathrm{d}\psi_i) = 0 \ , \tag{5.2}$$

for each index i. The modified Einstein's field equation are listed below. Again, as in section 3, the authors were able to solve the generalization of (3.7) provided that neither ϕ_i nor ψ_i depends on the radial coordinate. Then from (5.6) we have

$$\phi_i = \phi_{i0} + \phi_{i1} z \ , \ \psi_i = \psi_{i0} + \mathrm{e}^{\phi_i} \psi_{i1} \ , \tag{5.3}$$

which inserted into the (5.2) yields $\phi_{i1}\psi_{i1} = 0$. In (5.3) $\phi_{i0}, \phi_{i1}, \psi_{i0}, \psi_{i1}$ are integration constants. Thus the only non-trivial zero-order solution is given by

$$\phi_i^{(0)} = \phi_{i0} + \phi_{i1} z \ , \ \psi_i^{(0)} = \psi_{i0} \ . \tag{5.4}$$

The solution (3.10) and (3.11) remains unaffected provided ϕ_1^2 is replaced by $\Phi_1^2 = \sum \phi_{i1}^2$.

5.2 First order in ε

As a consequence of the presence of more massless scalar fields, one has to modify the equation (4.2) in the following manner. The equation (4.2b) becomes

$$\begin{aligned} &\frac{2}{M}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{M}\frac{\mathrm{d}\alpha}{\mathrm{d}r}\right] - \frac{4\varepsilon}{l^4}\mathrm{e}^{\phi+2\alpha}F_{\varphi z}^2 \\ &= -\frac{1}{M^2}\sum_i\left[\left(\frac{\partial\phi_i}{\partial r}\right)^2 + \mathrm{e}^{-2\phi_i}\left(\frac{\partial\psi_i}{\partial r}\right)^2\right] \ . \end{aligned} \tag{5.5}$$

The term $(\frac{\partial\phi}{\partial z})^2$ on the right-hand side of the equation (4.2e) should be replaced by

$$\sum_i\left[\left(\frac{\partial\phi_i}{\partial z}\right)^2 + \mathrm{e}^{-2\phi_i}\left(\frac{\partial\psi_i}{\partial z}\right)^2\right] \ . \tag{5.6}$$

Similarly the equation (4.2f) will be changed in an obvious way. Finally (4.2d) becomes

$$l^2\sum_i\left(\frac{\partial\phi_i}{\partial r}\frac{\partial\phi_i}{\partial z} + \mathrm{e}^{-2\phi_i}\frac{\partial\psi_i}{\partial r}\frac{\partial\psi_i}{\partial z}\right) = 4\varepsilon\mathrm{e}^{\phi}F_{r\varphi}F_{\varphi z} \ . \tag{5.7}$$

It was stated before that the scalar fields are decomposed into zero-order part and first-order correction as

$$\phi_i = \phi_{i0} + \phi_{i1} z + \varepsilon\phi_i^{(1)} \ , \ \psi_i = \psi_{i0} + \varepsilon\psi_i^{(1)} \ .$$

As a matter of fact it is seen that all terms in the modified Einstein's equations involving $(\frac{\partial\psi_i}{\partial r})^2, (\frac{\partial\psi_i}{\partial z})^2$ and even $(\frac{\partial\phi_i}{\partial r})^2$ are already of $\mathcal{O}(\varepsilon^2)$ and have to be ignored. Therefore we continue to keep our *Ansatz* (4.7) for the electromagnetic field. From this fact it immediately follows that the term $F \wedge F$ vanishes identically. The equation (4.9) should be modified due to the presence of the zero-order axionic field. But $\psi^{(0)} = \psi_0$ is constant, which can be set equal to one, and so in particular the equation (4.9) applies in this case as well. It also means that, for example, from (4.2d), the scalar fields ϕ_i are given by

$$\phi_i = \phi_{i0} + \tilde{\phi}_{i1} z \ , \tag{5.8}$$

with constants $\tilde{\phi}_{i1}$ including the ε-order corrections to ϕ_{i1}. The resulting metric and physical quantities are still given by (4.11) and (4.12) provided $\tilde{\phi}_1^2$ is replaced by $\tilde{\Phi}_1^2 = \sum \tilde{\phi}_{i1}^2$.

Since the Einstein's equations give us no information with respect to the fields ψ_i one has to solve their equations of motion (5.2). One can carry out the separation of variables to obtain

$$\psi_i = \psi_{i0} + \varepsilon e^{\phi_{i1} z}[A_i \cos(v_i z) + B_i \sin(v_i z)]\ \eta_i(r)\ , \tag{5.9}$$

where A_i, B_i and v_i are arbitrary constants and the functions η_i are solutions of second-order equations that can be transformed into the form

$$e^{-2\alpha} \frac{d}{dm} \left(l^2 \frac{d\eta_i}{dm} \right) - (\phi_{i1}^2 + v_i^2)\ \eta_i = 0\ . \tag{5.10}$$

6 String-inspired theory of gravity

The aim of this section is to show that the solution described by the metric (4.11) actually remains unaltered even if string-inspired charged gravity is taken under consideration [3].

The total string-inspired effective action (2.2) can be rewritten as [3, 11]

$$\begin{aligned} S_{\text{eff}} = \int_{\mathcal{M}} & [*R + 16\pi * \mu \\ & - \sum_i (d\phi_i \wedge * d\phi_i + e^{-2\phi_i} d\psi_i \wedge * d\psi_i) \\ & - 8\pi^2 \varepsilon e^{\phi} e(\mathcal{M}) + 4\varepsilon\ \psi\ \text{Tr}\ \Omega \wedge \Omega \\ & + 2\varepsilon\ e^{\phi}\ F \wedge *F - 4\varepsilon\ \psi\ F \wedge F]\ . \end{aligned} \tag{6.1}$$

In terms of the inverse string tension α' (Regge slope) and the string coupling constant g the parameter ε is expressed like $\varepsilon = \frac{\alpha'}{4g^2}$. The Euler class $e(\mathcal{M})$ of $T\mathcal{M}$ occuring in (6.1) is in four dimensions equal to [10]

$$e(\mathcal{M}) = \frac{1}{8\pi^2} \left(R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2 \right) \eta\ , \tag{6.2}$$

η is the volume element with components $\eta_{\alpha\beta\gamma\delta}$.

We have also added an extra contribution arising from a field μ. Physically it may represent an energy density of a fermionic matter, that is in our model approximated by a perfect fluid of a pressure p. Although this picture is rather intuitive and is not as transparent as in former EMD, later it will turn out to be useful. If one wants to supress the fermionic matter and recover an ordinary string-inspired action with the cosmological constant Λ, then our approach is fruitful too since the state equation $\mu + p = 0$ along with $\Lambda = -8\pi p$ gives the desired modification of the action. The fields $\phi_N \equiv \phi$ and $\psi_N \equiv \psi$ may be referred to as the dilaton and axion respectively.

For any cylindrically symmetric stationary metric, i.e. metric depending on the radial coordinate with the only non-vanishing cross-term $g_{\varphi t}$, a straightforward calculation gives the following useful formula for the Euler class (6.2)

$$\begin{aligned} e(\mathcal{M}) &= \frac{1}{4\pi^2}\frac{\mathrm{d}}{\mathrm{d}r}\left\{\left[\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 e^{2\alpha} + 2\frac{\mathrm{d}\alpha}{\mathrm{d}r}\frac{\mathrm{d}l^2}{\mathrm{d}r}\right] l^{-1}\mathrm{e}^{\alpha+\gamma-3\delta}\frac{\mathrm{d}\gamma}{\mathrm{d}r}\right\} \\ &\times \; l^{-1}\mathrm{e}^{-\alpha-\gamma-\delta}\,\Theta^{\hat{0}}\wedge\Theta^{\hat{1}}\wedge\Theta^{\hat{2}}\wedge\Theta^{\hat{3}} \; . \end{aligned} \tag{6.3}$$

In (6.3) the function γ is given by $g_{zz} = -\mathrm{e}^{2\gamma}$. Thus for (3.9) the Euler class vanishes. For the basis (2.1) it turns out that the term

$$\mathrm{Tr}\;\Omega\wedge\Omega = -\frac{1}{4}R^{\alpha\beta}{}_{\gamma\delta}R^{\sigma\tau}{}_{\alpha\beta}\;\eta^{\gamma\delta}{}_{\sigma\tau}\;\eta$$

is identically vanishing.

Since the magnetic field (4.7) is purely longitudinal, the term $F\wedge F$ is vanishing too and our problem in fact reduces to the Einstein-Maxwell-dilaton plus fluid system disscussed in previous sections. This completes the proof that the metric (4.11) after appropriate replacement $\tilde{\phi}_1^2 \to \tilde{\Phi}_1^2 = \sum\tilde{\phi}_{i1}^2$ constitutes string-inspired solution. The scalar fields are given by (5.8) and (5.9), subject to the equation (5.10).

7 On some atributes of the solution

We briefly comment on some physical atributes of the solution (4.11).

Of course no every specialization of the integration constants in (4.11) leads to cylindrically symmetric spacetime. If one requires the found solutions to be cylindrically symmetric and regular at the origin the axial symmetry condition and the elementary flatness condition have to be imposed [12]. In our case, provided $\alpha \propto r^2$ for small vaues of r, these conditions give

$$\begin{aligned} \Omega^2 - \frac{1}{4}\tilde{\phi}_1^2 - \frac{D}{2} &= \pm\, C \; , \\ \Omega^2 + \frac{1}{4}\tilde{\phi}_1^2 + E &= 0 \; , \\ F &= \frac{\Omega}{C} \; . \end{aligned} \tag{7.1}$$

Clearly the Lorentz force is vanishing. Also it can straightforwardly seen that the source term $A\wedge *j$ is exact form. Furthermore non-geodesic motion of the fluid should be understood as a mere consequence of a pressure inhomogeneity. Indeed it follows from fact that the accelaration is $\dot{u} = -\,\mathrm{d}\alpha$ and that the Bianchi identity, (3.5) or (4.10), can be rewritten as

$$(\mu+p)\,\frac{\mathrm{d}\alpha}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r} \; .$$

The vorticity 1-form ω according to (2.4) equals

$$\omega = \frac{1}{2} * (u\wedge\,\mathrm{d}u) = \Omega\mathrm{e}^{-2\alpha}\,\mathrm{d}z \; .$$

In this way the metric (4.11) is static if and only if $\Omega = 0$.

Let us also write down how the energy conditions restrict the ranges of the integration constants in (4.11). The strong along with the dominant energy condition imply the following two inequalities

$$\begin{aligned} D - 4\varepsilon B^2 \alpha + \frac{1}{2}\tilde{\phi}_1^2 e^{2\alpha} \geq 0 \ , \\ 4\varepsilon B^2 \geq \tilde{\phi}_1^2 e^{2\alpha} \ . \end{aligned} \tag{7.2}$$

Next remark concernes the algebraic classification of the Weyl tensor. It turns out that the metric (4.11) is of Petrov type D except on the hypersurfaces (one or more) given by

$$4\varepsilon B^2 \alpha = 2\varepsilon B^2 + D \ ,$$

where it is of type O.

Last remark clarifies the connection between (4.11) and Gödel-type solutions [4], provided that the function α and integration constants in (4.11) are chosen conviniently. The following appropriate choice respecting regularity conditions (7.1) has been done

$$\begin{aligned} \Omega^2 = \varepsilon B^2 + \frac{1}{2a^2} - \frac{1}{4}\tilde{\phi}_1^2 \ , \ E = -\left(\frac{1}{2a^2} + \varepsilon B^2\right) \ , \\ D = \frac{1}{a^2} + 2\varepsilon B^2 - \tilde{\phi}_1^2 + 2C \ , \ F = \frac{\Omega}{C} \ . \end{aligned} \tag{7.3}$$

The physical meaning of the constant a will be clear shortly. Let us now specify the arbitrary function α as $\alpha = 2a^2 C \mathrm{sh}^2(\frac{r}{2a})$. For simplicity let us consider only the dilatonic and axionic fields. We obtain new metric that depends upon three parameters: a, B and C (explicit form is omitted here). This solution describes an inhomogeneous universe and from (7.2) it is immediate that C must not be positive. If C is set equal to zero, the general solution (4.11) becomes

$$ds^2 = [dt + 4a^2\Omega \mathrm{sh}^2(\frac{r}{2a})\, d\varphi]^2 - a^2 \mathrm{sh}^2(\frac{r}{a})\, d\varphi^2 - dz^2 - dr^2, \tag{7.4}$$

which is manifestly of the Gödel-type. If in addition the state equation $p + \mu = 0$ is requested, the scalar field contribution must have the form

$$\tilde{\phi}_1^2 = \frac{1}{a^2} + 2\varepsilon B^2 \ . \tag{7.5}$$

The latter equation reflects fact that the non-vanishing cosmological constant rather then the perfect fluid is considered. If we let $\varepsilon \to 0$ then from the section 3, especially from the equation (3.8), one has $a^2 \phi_1^2 = 1$, and from (2.3), (5.8) and (7.5) the ε-order corrected dilaton is given by

$$\phi = \phi_0 + \phi_1 z \left(1 + \varepsilon B^2 a\right) \ . \tag{7.6}$$

The relationship between fundamental parameters of the theory becomes

$$4\Omega^2 - \frac{1}{a^2} = 2\varepsilon B^2 \ , \tag{7.7}$$

subject to the inequality $2\varepsilon a^2 B^2 \geq 1$, which follows from the energy conditions (7.2).

For the axion one has the equations (5.9) and (5.10). With help of the elementary theory of Legendre polynomials we find $v = \phi_1$ and

$$\psi = \psi_0 + \varepsilon \mathrm{e}^{\phi_1 z} \mathrm{ch}\left(\frac{r}{a}\right) \left[A \cos\left(\frac{z}{a}\right) + B \sin\left(\frac{z}{a}\right)\right] .$$

Note that in zero-order regime $\varepsilon \to 0$ one has $a^2\phi_1^2 = 1$ and $4\Omega^2 = a^{-2}$, the latter equality immediately implying $g_{\varphi\varphi} \leq 0$. Thus there are no CTC's in the spacetime. On the other hand in ε-order framework, since because of (7.2) is ε generally non-negative, $g_{\varphi\varphi}$ becomes positive for sufficiently large r. Thus the first order corrections cause the chronology violation.

Equations (7.4), (7.6) and (7.7) are identical with these of Kanti and Vayonakis [3] in string-inspired charged gravity framework, when $\varepsilon = \frac{\alpha'}{4g^2}$. Their α'-order solution arising from Som and Raychaudhuri *Ansatz* for the electromagnetic field turned out to be most favored between others cases, also belonging to the α'-order.

We could also obtain proper generalization of another solution in [3] with a positive cosmological constant, simply by carrying out the imaginary transformation $a \to ia$ in (7.3), but we will not follow this line further.

8 Discussion

In the paper a class of stationary symmetric spacetimes within the framework of Einstein-Maxwell-dilaton gravity was found, that exhibits cylindrical symmetry. This solution is exact up to first order in parameter ε. Provided that the scalar fields in zeroth order do not depend on the radial coordinate we were also able to find the generalization to the case when more scalar fields are present.

The solution obtained depends upon the dilaton gradient and the magnetic field in the longitudinal direction. It is worthwhile to note that the Gauss-Bonnet term vanishes. Namely for this reason is found metric exact α'-order solution in string-inspired theory framework.

Since the forms of zero and first order solutions are similar, we can straightforwardly find out what is a consequence of the electromagnetic field presence with respect to the chronology violation. It turns out that it has the power to break down the chronology even if the zero-order solution was chronologically well behaved.

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