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Prohlašuji, že jsem tuto práci vypracoval samostatně a použil pouze uvedené zdroje a literaturu.

Datum a podpis

Abstrakt

Obsahem této práce je souhrn studia $D0$-brán [1, 2, 3], jenž patří k fundamentálním objektům strunové teorie. Prvotním impulsem byl článek [4] zabývající se jejich chováním při nízkých energiích pouze v plochém prostoru. Z tohoto článku zopakujeme odvození Hamiltoniánu, na který poté aplikujeme numerickou metodu [5] k nalezení základního stavu. Během tohoto procesu jsme našli v článku [4] chybu. Naše numerické řešení v porovnání s odlišnou numerickou metodou popsanou v článku [6, 7, 8] poskytuje přesnější řešení. Výsledky byly publikovány v [9].

Dalším přirozeným krokem je studium zakřiveného superprostoru, kde mohou $D0$-brány netriviálně [10] interagovat s pozaďovými poli. V této době ještě nebyl znám supersymetrický popis této nejobecnější konfigurace při nízkých energiích a proto jsme se o něj chtěli pokusit. Mezitím řešení tohoto problému bylo zveřejněno článkem [11, 12, 13] a to bylo důvodem k zastavení našeho odvozování.

Uspořádání práce nerespektuje časovou posloupnost našich jednotlivých kroků při studiu. Začneme krátkým shrnutím článku [11] a poté uvidíme jak tyto výsledky odpovídají výchozímu bodu v článku [9]. Tímto postupem chceme docílit hlubšího pochopení dané problematiky.

Kapitola 1 popisuje naši konvenci a nezbytné matemtické zázemí, které se intenzivně využívá v přehledu článku [11] uvedeném v kapitole 2. Konkrétně se budeme zabývat superprostory [14] a supervnořením [15, 16, 17, 18, 19, 3, 20]. Hlavním výsledkem jsou pohybové rovnice pro $D0$-brány *(uvedeme obecné řešení $D0$-brány v plochém prostoru)*, které jsou ekvivalentní rovnicím získaným dimensionální redukcí supersymetrické Yang-Millsovi teorie [4]. Tyto rovnice jsou dále kvantovány a řešeny v článku [9], který je přiložen a jehož přehled obsahuje poslední kapitola.

Abstract

We review the work done on the $D0$-branes [1, 2, 3] which are fundamental objects of the string theory. The starting point of this study was inspired by the paper [4] which is focused on their low-energy limit in flat space. We repeat the derivation of the Hamiltonian in the paper on which we apply a numerical method [5] to find a ground-state wave function. We also found a mistake in the paper [4] during this process. Our solution seems to be more accurate compared to another numerical method [6, 7, 8]. The result was published in the paper [9].

A natural next step is to explore curved superspace where the $D0$-branes can interact with the supergravity background fields [10] in a non trivial way. There were no supersymmetric description at the low-energy limit of this most general configuration thus we wanted to attempt to create it. Meanwhile the solution of this problem is published in the paper [11, 12, 13] and we halted our derivation.

The presentation in this thesis is not chronological. We begin with a summary of the paper [11] and then we see how its result corresponds to the starting point of our paper [9]. The purpose of this approach is to give a deeper understanding of the whole picture.

The section 1 describes our convention and necessary mathematical calculus extensively used in the summary of the paper [11] which is given in the section 2. Namely, we deal with curved superspaces [14] and the superembedding [15, 16, 17, 18, 19, 3, 20] approach. The main result from this section is the $D0$-brane equation of motion *(we give a complete $D0$-brane solution in flat space)* which are equivalent to the ones that came up from a dimensional reduction of a supersymmetric Yang-Mills theory [4]. These equations are quantized and solved in the attached paper [9]. Its summary is in the last section.

Contents

1 Mathematical Fundamentals

1.1 Notation

We use following indices and symbols in the paper.

$m, n, \ \ldots$	bosonic coordinate indices
$a, b, \ \ldots$	bosonic frame, noncoordinate, indices
$\mu, \nu, \ \ldots$	fermionic coordinate indices
$\alpha, \beta, \ \ldots$	fermionic frame, noncoordinate, indices
$M, N, \ \ldots$	superspace coordinate indices, *i.e.* $M = (m, \mu)$
$A, B, \ \ldots$	superspace frame, noncoordinate, indices, *i.e.* $A = (a, \alpha)$
$i, j, \ \ldots$	vector indices transforming in group $SO(n)$
$\rho, \sigma, \ \ldots$	gauge indices
$\hat{\Omega} \ \ldots$	pullback of the form Ω

We strictly underline all target superspace indices in the section 2 to distinguish them from non underlined worldvolume ones if not explicitly stated otherwise.

1.2 Space Geometry

There is a nice geometrical description of the curved superspaces which is coordinate free notation. We start with the more familiar purely bosonic space where we illustrate this approach and then we add the fermionic part to generalize it to superspace. Finally the non chiral IIA supergravity is discussed.

Let us have a manifold with local coordinates X^m. The indices $m, n, \ldots$ are strictly used only for coordinate basis. We use vielbeins $E_m{}^a(X)$ to define a new frame

$$E^a = dX^m E_m{}^a$$

where the dX^m are the basic forms in the coordinate basis and the indices $a, b, \ldots$ labels the frame vectors. The matrix $E_m{}^a$ is invertible and its inverse $E_a{}^m$ satisfies

$$E_m{}^a E_a{}^n = \delta_m{}^n \ , \qquad E_a{}^n E_n{}^b = \delta_a{}^b \ .$$

This inverse generates new corresponding basis tangent vectors

$$E_a = E_a{}^n \partial_n$$

which complete the basic building block for the following geometry description.

The vielbeins allows to write an arbitrary metric $g_{mn} dX^m \otimes dX^n = g_{ab} E^a \otimes E^b$ where the g_{ab} is diagonal. This is in general not possible in a coordinate basis. This fact helps to introduce spinors in curved space for example.

The external derivative on wedge product is defined as

$$d(\Omega \wedge \Theta) = \Omega \wedge d\Theta + (-)^p d\Omega \wedge \Theta$$

for arbitrary form Ω and p-form Θ.

To describe the geometry of the space we introduce the covariant derivative acting on covariant V^a and contravariant V_a vectors as

$$\nabla V^a = dV^a + V^b \wedge \omega_b{}^a \; ; \qquad \nabla V_a = dV_a - \omega_a{}^b \wedge V_b \tag{1.1}$$

where the $\omega_a{}^b \equiv E^c \omega_{ca}{}^b$ is a connection form [25, 27] generating the corresponding torsion $T_{bc}{}^a$ and curvature $R_{dca}{}^b$ which define the following two forms

$$\begin{aligned} T^a &\equiv \frac{1}{2} E^c \wedge E^b T_{bc}{}^a \\ R_a{}^b &\equiv \frac{1}{2} E^c \wedge E^d R_{dca}{}^b \end{aligned}$$

satisfying

$$\begin{aligned} T^a &= \nabla E^a \\ R_a{}^b &= d\omega_a{}^b + \omega_a{}^c \omega_c{}^b \end{aligned}$$

together with Bianchi identities

$$\begin{aligned} \nabla T^a &\equiv E^b \wedge R_b{}^a \\ \nabla R_a{}^b &\equiv 0 \,. \end{aligned}$$

The connection above can be identified with a connection in the coordinate basis. These connections generate the same parallel transport for a given vector field $V^a E_a = V^n \partial_n$. Moreover if the covariant coordinate derivative is compatible with the metric

$$\nabla g = 0$$

the ω^{ab} is antisymmetric in the indices.

Let us illustrate the above purely mathematical description on something physical. We pick up a Riemannian manifold to see what happens. We can always choose vielbeins to diagonalize the metric

$$E_a{}^m E_b{}^n g_{mn} = \eta_{ab} \,.$$

A structure group G acts on the frame bundle. Clearly the Lorentz group does not change the right hand side of the equation above. We can now rewrite the covariant derivative (1.1) in the following compact form

$$\nabla_a \equiv E_a + \frac{1}{2} \omega_{ab}{}^c \mathcal{G}_c{}^b$$

where the matrices $\mathcal{G}_c{}^b$ are generators of the Lorentz group vector representation for the covariant or contravariant vectors. The covariant derivative algebra reads

$$[\nabla_a, \nabla_b] = -T_{ab}{}^c \nabla_c + \frac{1}{2} R_{abc}{}^d \mathcal{G}_d{}^c$$

and fully determine the geometry. This is the coordinate free notation.

1.3 Superspace

It is convenient to work with superspace in order to create a manifestly supersymmetric theories. Superspace is a manifold which has extra fermionic anticommuting variables θ^μ, *i.e.* the local coordinates reads $Z^M = (X^m, \theta^\mu)$. Similarly to the previous convention the indices $M, N, ...$ and $\mu, \nu, ...$ are strictly used only for coordinate basis. An arbitrary form dZ^M has grade, the bosonic $|dZ^m| = 0$ and the fermionic $|dZ^\mu| = 1$. The graded wedge product between the superforms is defined as

$$dZ^M \wedge dZ^N = -(-)^{MN} dZ^M \wedge dZ^N$$

to reflect the anticommuting property of the fermionic coordinate and where the exponent MN is the product of the form grades. We use again vielbeins $E_M{}^A(Z)$ or rather supervielbeins to define superinvariant frames

$$E^A = dZ^M E_M{}^A$$

where the dZ^M are basic forms in the coordinate basis. The indices $A, B, \ldots$ containing also fermionic part $\alpha, \beta, ...$ solely labels the frame vectors. The matrix $E_M{}^A$ is invertible and its inverse $E_A{}^M$ satisfies

$$E_M{}^A E_A{}^N = \delta_M{}^N \ , \qquad E_A{}^N E_N{}^B = \delta_A{}^B \ .$$

This inverse generates the corresponding basis tangent vectors

$$E_A = E_A{}^N \partial_N \ . \tag{1.2}$$

Based on our previous discussion we can directly write the covariant derivative

$$\nabla_A \equiv E_A + \frac{1}{2}\omega_{AB}{}^C \mathcal{G}_C{}^B \tag{1.3}$$

where the $\omega_A{}^B \equiv E^C \omega_{CA}{}^B$ is a connection superform and the $\mathcal{G}_C{}^B$ are now appropriate generators of a structure supergoup G. The corresponding algebra

$$[\nabla_A, \nabla_B\} = -T_{AB}{}^C \nabla_C + \frac{1}{2} R_{ABC}{}^D \mathcal{G}_D{}^C$$

contains the symbol $[,\}$ which is the graded commutator. The torsion and curvature

$$\begin{aligned} T^A &\equiv \frac{1}{2} E^C \wedge E^B T_{BC}{}^A \\ R_A{}^B &\equiv \frac{1}{2} E^C \wedge E^D R_{DCA}{}^B \end{aligned}$$

satisfy again

$$\begin{aligned} T^A &= \nabla E^A \\ R_A{}^B &= d\omega_A{}^B + \omega_A{}^C \omega_C{}^B \end{aligned} \tag{1.4}$$

together with the Bianchi identities

$$\begin{aligned} \nabla T^A &\equiv E^B \wedge R_B{}^A \\ \nabla R_A{}^B &\equiv 0 \ . \end{aligned} \tag{1.5}$$

1.4 Supergravity

We now restrict the structure supergroup G on the superspace [21, 22, 23, 24, 25] and see the implication of this selection. Then we apply the formalism to the IIA supergravity.

We choose the Lorentz group. This choice simplifies the covariant derivative (1.3) and moreover the two in general independent Bianchi identities (1.5) are equivalent under this choice [26]. An infinitesimal transformation under the Lorentz group $\delta L_B{}^A$ changes the vector V^A by

$$\delta V^A = V^B \delta L_B{}^A \tag{1.6}$$

where

$$\delta L_a{}^\alpha = 0 \ , \qquad \delta L_\alpha{}^a = 0 \ . \tag{1.7}$$

Thus there is no mixing between fermionic and bosonic indices. In fact the V^a transforms in the vector representation and the V^α in a spinor representation of the Lorentz group. The connection superform $\omega_B{}^A$ in (1.3) and also the curvature superform (1.4) has the same structure as $\delta L_A{}^B$ in (1.7). This property is of course preserved when performing Lorentz transformation in the tangent superspace [25]

$$\delta\omega_A{}^B = \omega_A{}^C \delta L_C{}^B - \delta L_A{}^C \omega_C{}^B - d\omega_A{}^B \ .$$

This allows us to write the covariant derivative (1.3) in the following form

$$\nabla_A = E_A + \frac{1}{2}\omega_{Aa}{}^b \mathcal{G}_b{}^a \tag{1.8}$$

where the $\mathcal{G}_b{}^a$ are the Lorentz algebra generators in an appropriate representation. The algebra of the supercovariant derivatives is

$$[\nabla_A, \nabla_B\} = -T_{AB}{}^C \nabla_C + \frac{1}{2} R_{ABc}{}^d \mathcal{G}_d{}^c \ .$$

1.5 IIA

Let us take a closer look at the IIA nonchiral superspace which is locally parametrized by coordinates $Z^M = (X^m, \theta^\mu, \theta^{\dot\mu})$ where $m = 0, \ldots, 9$ and $\mu, \dot\mu = 1, \ldots, 16$. An arbitrary tangent vector $V = V^M \partial_M$ can be written in the basis (1.2) generated by the vielbeins $E_A{}^M$

$$V = V^A E_A = V^a E_a + V^\alpha E_\alpha + V^{\dot\alpha} E_{\dot\alpha} \ . \tag{1.9}$$

The V^a transforms in the vector representation of the Lorentz group $SO(1,9)$ and the fermionic part V^α is the chiral spinor in the **16** and $V^{\dot\alpha}$ is an antichiral spinor in the $\underline{\mathbf{16}}$. Please see the Appendix for the spinor notation.

The IIA superspace is described by a supervielbein $E_A{}^M$, a connection superform $\omega_b{}^c$, a gauge superform A, a 2-superform B and a 3-superform Γ [25, 27]. The super covariant derivative reads

$$\nabla_A = E_A + \frac{1}{2}\omega_{Aa}{}^b\,(\mathcal{G}_b{}^a) + A_A\mathcal{Z} \tag{1.10}$$

where the $\mathcal{Z}$ is the gauge central charge. Their super algebra is

$$[\nabla_A, \nabla_B\} = -T_{AB}{}^C\nabla_C + \frac{1}{2}R_{ABc}{}^d\,(\mathcal{G}_d{}^c) + F_{AB}\mathcal{Z}\ . \tag{1.11}$$

The gauge field strength

$$F = \frac{1}{2}E^B \wedge E^A F_{AB}$$

defined by the relation

$$F = dA - A \wedge A \tag{1.12}$$

satisfies Bianchi identity

$$\nabla F = 0\ . \tag{1.13}$$

The Bianchi identities (1.5) are not changed for the new super covariant derivative (1.10) as the torsion and curvature super forms (1.4) are singlets under the gauge group.

The trivial solution of all Bianchi identities [23] is the flat superspace where all the background fields vanish. In spite of the fact that $R = 0$ there are non zero components of the torsion

$$T_{\alpha\beta}{}^a = -i\sigma^a{}_{\alpha\beta}\ , \qquad T_{\dot\alpha\dot\beta}{}^a = -i\tilde\sigma^a{}_{\dot\alpha\dot\beta} \tag{1.14}$$

which cannot be gauged away. Please take a look at the Appendix for the definition of the σ matrices (A.2).

2 Superembedding Approach

2.1 Superembedding Strategy

We show how a simple geometrical embedding condition generates the equation of motion for the $D0$-brane. This procedure can be done for arbitrary Dp-branes but to obtain the equation of motion is not guaranteed in general [11]. Also, the equation of motion for multiple $D0$-branes in the low-energy limit is derived. In fact we repeat the derivation from the paper [11] and add a complete $D0$-brane solution in flat superspace.

The superembedding which describes the Dp-brane is the map between two superspaces. The first one is a worldvolume and the other is a target superspace. The target superspace for p even is [1, 28] the above described IIA superspace containing the mentioned odd superforms.

The worldvolume bosonic dimension is $p+1$ while the fermionic dimension is half of the target superspace fermionic dimension because the presence of the Dp-branes breaks half of the supersymmetries due to kappa symmetry [29]. We underline the target space indices to clearly distinguish them from not underlined worldvolume ones.

The superembedding

$$Z^{\underline{M}}(Z) \tag{2.15}$$

generates a pullback of the form $E^{\underline{A}}$

$$\hat{E}^{\underline{A}} \equiv E^B \hat{E}_B{}^{\underline{A}} \tag{2.16}$$

given by the superembedding matrix

$$\hat{E}_B{}^{\underline{A}} = E_B{}^M \partial_M Z^{\underline{M}}(Z) E_{\underline{M}}{}^{\underline{A}} \, . \tag{2.17}$$

The simple geometrical embedding condition mentioned in this section preamble is

$$\hat{E}_\alpha{}^{\underline{a}} = 0 \tag{2.18}$$

which tells us that the pullback of the bosonic form $E^{\underline{a}}$ is also only bosonic in the worldvolume superspace. The Dp-brane described by the superembedding (2.15) naturally splits the tangent bundle spanned by the frame $E_{\underline{a}}$ into two linearly independent parts $\underline{T} \oplus \underline{N}$. The $\underline{T}$ is the tangent bundle spanned by the pushforwarded frame E_a and the $\underline{N}$ is the normal bundle *(its vectors are perpendicular to the $\underline{T}$)*. To have an elegant description of this split we transform the basis $E_{\underline{a}}$ in the following manner. The transformation matrix $u_{\underline{a}}{}^r(Z)$ satisfying

$$u_{\underline{a}}{}^r u_r{}^{\underline{b}} = \delta_{\underline{a}}{}^{\underline{b}} \, , \qquad u_r{}^{\underline{a}} u_{\underline{a}}{}^s = \delta_r{}^s \tag{2.19}$$

where $r \equiv (a, i)$ is a composed index is chosen in such way the $E_a \equiv u_a{}^{\underline{a}} E_{\underline{a}}$ are basis vectors of the $\underline{T}$ and $E_i \equiv u_i{}^{\underline{a}} E_{\underline{a}}$ span the perpendicular complement $\underline{N}$ as discussed above.

Thus an arbitrary tangent vector $V = V^{\underline{a}} E_{\underline{a}}$ can be always written as $V = V^a E_a + V^i E_i$. We see the original target tangent space symmetry $SO(1,9)$ reduces to $SO(1,p) \otimes SO(9-p)$ when the Dp-brane is present. This is the reason why we exceptionally omitted the underline for the superspace index r because its first components a transform exactly under the same vector representation of the $SO(1,p)$ as the worldvolume indices a and the index i is the new one transforming under the $SO(9-p)$.

The matrix $u_{\underline{a}}{}^r(Z)$ generates the new forms on the cotangent bundle in the target superspace

$$E^r \equiv E^{\underline{a}} u_{\underline{a}}{}^r \tag{2.20}$$

where their pullback satisfies

$$\hat{E}^i = 0 \;, \qquad \hat{E}^a = e^a \tag{2.21}$$

and also transform [30] the target superspace connection $\Omega_{\underline{b}}{}^{\underline{a}}$ *(we use the big* Ω *for the target space in contrast with* ω *for worldvolume)* to the new one

$$\Omega_r{}^s = du_r{}^{\underline{a}} u_{\underline{a}}{}^s + u_r{}^{\underline{b}} \Omega_{\underline{b}}{}^{\underline{a}} \; u_{\underline{a}}{}^s = \nabla u_r{}^{\underline{a}} u_{\underline{a}}{}^s \;. \tag{2.22}$$

The curvature superform is similarly changed to

$$R_r{}^s = u_r{}^{\underline{a}} R_{\underline{a}}{}^{\underline{b}} u_{\underline{b}}{}^s \;.$$

2.2 Induced geometry

The induced geometry on the worldwolume is discussed here. The pullback

$$\hat{E}^{\underline{a}} = e^a u_a{}^{\underline{a}}$$

can be derived from the definitions (2.20,2.21). We demand to have induced torsion T^a on the worldsuperspace [31]

$$\hat{T}^{\underline{a}} u_{\underline{a}}{}^b \equiv T^b = de^b + e^c \wedge \omega_c{}^b \;. \tag{2.23}$$

This is satisfied if the connection superform on the worldvolume is

$$\omega_c{}^b = du_c{}^{\underline{a}}(Z) u_{\underline{a}}{}^b + u_c{}^{\underline{b}} \hat{\Omega}_{\underline{b}}{}^{\underline{a}} \; u_{\underline{a}}{}^b \;. \tag{2.24}$$

Compared to the (2.22) this relation reads

$$\omega_c{}^b = \hat{\Omega}_c{}^b \;,$$

i.e. this worldvolume connection superform is equal to the pullback of the adapted target superspace connection superform. However the solution (2.24) is not unique. Let us write the bosonic part of the superform $\omega_c{}^b$ explicitly and insert it to the (2.23). We see the last term $e^c \wedge e^a \omega_{ac}{}^b$ is not changed when we add to the $\omega_c{}^b$ another form $e^a \phi_{ac}{}^b$ where its components $\phi_{ac}{}^b$ are symmetric in the lower indices.

This superconnection form implies the worldvolume curvature (1.4)

$$r_a{}^b = u_a{}^{\underline{a}} \hat{R}_{\underline{a}}{}^{\underline{b}} u_{\underline{b}}{}^b - \hat{\Omega}_a{}^i \wedge \ \hat{\Omega}_i{}^b \ .$$

We are interested in the $\hat{\Omega}_a{}^i$ because this superform plays crucial role in equation of motion derivation as we will see. There is useful relation between $\hat{\Omega}_a{}^i$ and the target superspace torsion $T^{\underline{a}}$. It can be found from the pullback of the following relation

$$\nabla E^i = E^{\underline{a}} \wedge \nabla u_{\underline{a}}{}^i + T^{\underline{a}} u_{\underline{a}}{}^i \tag{2.25}$$

and the result is

$$0 = -e^a \wedge \hat{\Omega}_a{}^i + \hat{T}^{\underline{a}} u_{\underline{a}}{}^i \ . \tag{2.26}$$

2.3 The $D0$-brane Case

We focus on the $D0$-brane from this moment. The other higher dimensional Dp-branes are considered in [11]. The equation of motion is derived here.

The world superspace parametrized by $Z^M = (x^0, \eta^\nu)$ has only one bosonic coordinate and $\nu = 1, \ldots, 16$. Remember that this is due to the fact the Dp-brane breaks half of the supersymmetries and the target superspace has two 16 dimensional spinors. Superinvariant frames are

$$\begin{aligned} e^0 &= dx^0 - i e^\alpha (\sigma^0)_{\alpha\beta} \eta^\beta \\ e^\alpha &= d\eta^\nu \delta_\nu{}^\alpha \end{aligned}$$

where the $\eta^\beta = \eta^\nu \delta_\nu{}^\alpha$ is defined. The supervielbeins $E_M{}^A$ can be directly read from the forms above.

To calculate the important superform $\hat{\Omega}_0{}^i$ from the equation (2.26) we have to know the target superspace superform $T^{\underline{a}}$. The general solution [23, 24] of the (1.5) for our target superspace is

$$T^{\underline{a}} = -\frac{1}{2} i E^{\underline{\alpha}} \wedge E^{\underline{\beta}} (\sigma^{\underline{a}})_{\underline{\alpha\beta}} - \ \frac{1}{2} i E^{\underline{\dot{\alpha}}} \wedge E^{\underline{\dot{\beta}}} \ (\tilde{\sigma}^{\underline{a}})_{\underline{\dot{\alpha}\dot{\beta}}} \ .$$

To go ahead we choose the pullback of the fermionic form. We can always choose coordinates where

$$\hat{E}^{\underline{\alpha}} \equiv e^\alpha \ , \qquad \hat{E}^{\underline{\dot{\alpha}}} = \ e^\beta h_\beta{}^{\underline{\dot{\alpha}}} + e^0 \chi^{\underline{\dot{\alpha}}} \ . \tag{2.27}$$

We are ready to insert all the necessary data into the equation (2.26). The term proportional to $e^\alpha \wedge e^\beta$ gives us

$$0 = (\sigma^{\underline{a}})_{\alpha\beta} u_{\underline{a}}{}^i + (h \sigma^{\underline{a}} h^T)_{\alpha\beta} u_{\underline{a}}{}^i \tag{2.28}$$

and for $e^0 \wedge e^\alpha$ we have

$$0 = (\hat{\Omega}_\gamma)_0{}^i - i(h\tilde{\sigma}^{\underline{a}})_{\gamma\underline{\dot{\alpha}}}\chi^{\underline{\dot{\alpha}}}u_{\underline{a}}{}^i \ .$$

Let us take a closer look at the connection form $\hat{\Omega}_0{}^i$ where we based on the previous equation can write

$$(\hat{\Omega})_0{}^i = ie^\gamma(h\tilde{\sigma}^{\underline{a}})_{\gamma\underline{\dot{\alpha}}}\ \chi^{\underline{\dot{\alpha}}}u_{\underline{a}}{}^i + e^0(\hat{\Omega}_0)_0{}^i \ .$$

When we compare equations (2.16,2.18,2.21) we get

$$\hat{E}^{\underline{a}} = e^0 u_0{}^{\underline{a}} = e^0 \hat{E}_0{}^{\underline{a}} \tag{2.29}$$

and from (2.22) we obtain

$$(\hat{\Omega})_0{}^i = d\hat{E}_0{}^{\underline{a}}u_{\underline{a}}{}^i + \hat{E}_0{}^{\underline{b}}\hat{\Omega}_{\underline{b}}{}^{\underline{a}}u_{\underline{a}}{}^i \ . \tag{2.30}$$

This relation can be thought of as the equation of motion for the $D0$-brane because it contains the second derivative of the embedding (2.15). We have to be very careful here as not all solution of this equation are physical. The physical solution must satisfy the superembedding condition (2.18)

$$\hat{E}_\alpha{}^{\underline{a}} = 0 \ .$$

The problem now is to calculate the left hand side of the equation of motion. The still unknown bosonic $h_\beta{}^{\underline{\dot{\alpha}}}$ function in the (2.30) can be found with help of target superspace background 3-superform H. Its pullback

$$\hat{H} = 0$$

for a $D0$-brane [11]. The general H in IIA superspace is [23, 24]

$$H = \frac{1}{2}iE^{\underline{a}} \wedge E^{\underline{\alpha}} \wedge E^{\underline{\beta}}\ (\sigma_{\underline{a}})_{\underline{\alpha}\underline{\beta}} - \ \frac{1}{2}iE^{\underline{a}} \wedge E^{\underline{\dot{\alpha}}} \wedge E^{\underline{\dot{\beta}}}\ (\tilde{\sigma}_{\underline{a}})_{\underline{\dot{\alpha}}\underline{\dot{\beta}}} + \ldots$$

where the dots means wedge products which contains at least two bosonic $E^{\underline{a}}$ and such terms will not contribute to the pullback for the $D0$-brane. The terms in $\hat{H}$ proportional to $e^0 \wedge e^\alpha \wedge e^\beta$ implies

$$0 = u_a{}^{\underline{a}}(\sigma_{\underline{a}})_{\alpha\beta} - u_a{}^{\underline{a}}\ (h\tilde{\sigma}_{\underline{a}}h^T)_{\alpha\beta} \ . \tag{2.31}$$

We now have two equations (2.28,2.31) for $h_\beta{}^{\underline{\dot{\alpha}}}$. There is a unique solution to these equations

$$h_\beta{}^{\underline{\dot{\alpha}}} = (\sigma^{\underline{b}})_{\underline{\beta}}{}^{\underline{\dot{\alpha}}}u_{\underline{b}}{}^0 = (\sigma^{\underline{b}})_{\underline{\beta}\underline{\alpha}}u_{\underline{b}}{}^0 \ ,$$

where the lowering of the dotted index is explained in the Appendix.

Now the unknown fermionic function $\chi^{\underline{\dot\alpha}}$ in the (2.30) can be solved from the torsion target space superforms. We can insert the de^α obtained from

$$\hat T^{\underline{\alpha}} \;=\; de^\alpha + e^\beta \wedge \hat\Omega_\beta{}^\alpha \qquad (2.32)$$

to the $\hat T^{\underline{\dot\alpha}}$ and the result is

$$\begin{aligned}
\hat T^{\underline{\dot\alpha}} &= d(e^\beta h_\beta{}^{\underline{\dot\alpha}} + e^0\chi^{\underline{\dot\alpha}}) + (e^\gamma h_\gamma{}^{\underline{\dot\beta}} + e^0\chi^{\underline{\dot\beta}}) \wedge \hat\Omega_{\underline{\dot\beta}}{}^{\underline{\dot\alpha}} \\
&= \hat T^\beta h_\beta{}^{\underline{\dot\alpha}} + e^\beta \wedge (dh_\beta{}^{\underline{\dot\alpha}} + h_\beta{}^{\underline{\dot\beta}}\hat\Omega_{\underline{\dot\beta}}{}^{\underline{\dot\alpha}} - \hat\Omega_\beta{}^\gamma h_\gamma{}^{\underline{\dot\alpha}}) + T^0\chi^{\underline{\dot\alpha}} + e^0 \wedge (d\chi^{\underline{\dot\alpha}} + \chi^{\underline{\dot\beta}}\hat\Omega_\beta{}^{\underline{\alpha}}) \\
&\equiv \hat T^\beta h_\beta{}^{\underline{\dot\alpha}} + e^\beta \wedge Dh_\beta{}^{\underline{\dot\alpha}} + T^0\chi^{\underline{\dot\alpha}} + e^0 \wedge D\chi^{\underline{\dot\alpha}} \qquad (2.33)
\end{aligned}$$

where we have used the (1.4,2.27) and defined the covariant derivative D. It can be shown that *(see the Appendix)*

$$Dh_\beta{}^{\underline{\dot\alpha}} = -(\sigma^{\underline{b}})_\beta{}^{\underline{\dot\alpha}} u_{\underline{b}}{}^i \hat\Omega_i{}^0 \; . \qquad (2.34)$$

The $\chi^{\underline{\dot\alpha}}$ can be solved [11] from the (2.33) for a target superspace background [23, 24].

To simplify the notation for the following sections we define matrices

$$(\sigma^0)_{\underline{\beta}\underline{\alpha}} \equiv (\sigma^{\underline{b}})_{\underline{\beta}\underline{\alpha}} u_{\underline{b}}{}^0 \; , \qquad (\sigma^i)_{\underline{\beta}\underline{\alpha}} \equiv (\sigma^{\underline{b}})_{\underline{\beta}\underline{\alpha}} u_{\underline{b}}{}^i \qquad (2.35)$$

and similarly for the $\tilde\sigma$. For example we can rewrite the (2.34) in the simple form

$$D(\sigma^0)_\beta{}^{\underline{\dot\alpha}} = -(\sigma^i)_\beta{}^{\underline{\dot\alpha}} \hat\Omega_i{}^0$$

and there is also a relation for the σ^i

$$D(\sigma^i)_\beta{}^{\underline{\dot\alpha}} + (\sigma^j)_\beta{}^{\underline{\dot\alpha}} \hat\Omega_j{}^i = -(\sigma^0)_\beta{}^{\underline{\dot\alpha}} \hat\Omega_0{}^i \; .$$

2.4 $D0$-brane Case Solution in Flat Space

Let us take a closer look at the simplest case, the flat *IIA* target superspace. We solve the equation of motion (2.30) and, as highlighted previously, select only solution which are physical, *i.e.* fulfil the superembedding condition (2.18).

The flat *IIA* superspace has torsion (1.14) superform $T^{\underline{a}}$ nonzero and the rest vanish. Moreover we choose coordinates in which the connection $\Omega_{\underline{B}}{}^{\underline{A}} = 0$.

This simplifies the relation (2.33) and the part proportional to the $e^\alpha \wedge e^\beta$ gives us $\chi^{\underline{\dot\alpha}} = 0$. Consequently the terms $e^0 \wedge e^\beta$ implies

$$\hat\Omega_0{}^i = 0 \; .$$

The equation of motion reads

$$0 = d\hat E_0{}^{\underline{a}} u_{\underline{a}}{}^i \qquad (2.36)$$

where $\hat{E}_0{}^{\underline{a}}$ is given by (2.17). To simplify the following steps we choose the transformation matrix $u_{\underline{a}}{}^i(Z)$ to be constant. The most general solution of equation (2.36) is

$$X^i = C^i x^0 + D^i(\eta) \tag{2.37}$$

where we have defined the $X^i \equiv X^{\underline{m}} \delta_{\underline{m}}{}^{\underline{a}} u_{\underline{a}}{}^i$, C^i is a constant and $D^i(\eta)$ is an arbitrary function of the fermionic coordinates. We now show that this solution is not physical in general. Before we insert this solution into the superembedding condition (2.18) we need to write the rest of the superembedding corresponding to our pullback choice (2.21)

$$\begin{aligned} X^{\underline{m}} &= x^0 u_0{}^{\underline{a}} \delta_{\underline{a}}{}^{\underline{m}} + F^{\underline{m}}(\eta) \\ \theta^{\underline{\mu}} &= \eta^\mu \\ \theta^{\underline{\dot{\mu}}} &= h_\mu{}^{\underline{\dot{\mu}}} \eta^\mu \end{aligned} \tag{2.38}$$

where the arbitrary function $F^{\underline{m}}(\eta)$ of the fermionic coordinates are related to the previous $D^i(\eta)$ through the X^i definition above. This superembedding induces the right torsion (2.23). The superembedding condition restrict the most general solution (2.37) only to the

$$X^i = D^i$$

where the D^i is constant. This is the only physical solution. The only static $D0$-brane could surely not be the case in another general coordinate. To have solutions representing $D0$-brane moving with constant speed we have to look at the equivalent solution in the $X^{\underline{m}}$ coordinate

$$X^{\underline{m}} = x^0 u_0{}^{\underline{a}} \delta_{\underline{a}}{}^{\underline{m}} + F^{\underline{m}} \tag{2.39}$$

where $F^{\underline{m}}$ is constant.

It would also be very interesting to try to find similar solutions for other simple superspaces, for instance for the AdS-like ones [32].

2.5 Multiple $D0$-branes

We know the equation of motion (2.36) for one $D0$-brane. A derivation of a similar equation for multiple $D0$-branes is not known yet. However there is the method given in [11] how to find the low-energy limit in the *IIA* supergravity target space. The main idea is to have a process which can be easily generalized to the curved superspace and which for the flat one generates results that one gets from super Yang-Mills theory with gauge group $SU(N)$ [33]. We show here only the flat superspace derivation just to confirm the correspondence.

To describe multiple $D0$-branes we introduce non-Abelian gauge $SU(N)$ connection superform on the worldvolume

$$A = e^0 A_0 + e^\alpha A_\alpha$$

with the field strength (1.12)

$$F=\frac{1}{2}e^{\alpha}\wedge e^{\beta}F_{\alpha\beta}+e^{\alpha}\wedge e^{0}F_{0\beta}$$

satisfying the Bianchi identity (1.13) $\nabla F=0$ which in purely fermionic components reads

$$\nabla_{(\alpha}F_{\beta\gamma)}+T_{(\alpha\beta}{}^{0}F_{0\gamma)}=0 \tag{2.40}$$

and in one bosonic

$$\nabla_{0}F_{\alpha\beta}-\nabla_{\beta}F_{0\alpha}-\nabla_{\alpha}F_{0\beta}=0\ . \tag{2.41}$$

The low-energy system of the multiple $D0$-branes should be described by a $SU(N)$ gauge valued field X_i transforming in the vector representation of the $SO(9)$. We have a natural choice (2.35) how to compose this term to the field strength

$$F_{\alpha\beta}=2i(\sigma^{i})_{\alpha\beta}X_{i}\ .$$

The induced torsion has to satisfy (2.23) and for our target torsion (1.14) reads

$$T_{\alpha\beta}{}^{0}=-2i(\sigma^{0})_{\alpha\beta}\ .$$

Inserting the field strength antsatz into the Bianchi identity (2.40)

$$2i(\sigma^{i})_{(\beta\gamma}\nabla_{\alpha)}X_{i}-2i(\sigma^{0})_{(\alpha\beta}F_{0\gamma)}=0 \tag{2.42}$$

and demanding further relations

$$\begin{aligned}\nabla_{\alpha}X_{i}&=i(\sigma_{i})_{\alpha\delta}\Psi^{\delta}\\ F_{0\gamma}&=-i(\sigma_{0})_{\gamma\delta}\Psi^{\delta}\end{aligned}$$

we see the left hand side of the (2.42) reduces to the Fierz Identity (A.4) and thus the Bianchi identity is satisfied. Let us try to insert the relations above into the second Bianchi identity (2.41) which now reads

$$2i(\sigma^{i})_{\alpha\beta}\nabla_{0}X_{i}=-(\sigma_{0})_{\alpha\delta}\nabla_{\beta}\Psi^{\delta}-(\sigma_{0})_{\beta\delta}\nabla_{\alpha}\Psi^{\delta}\ .$$

We can solve the $\nabla_{\alpha}\Psi^{\delta}$ from this Bianchi identity

$$\nabla_{\alpha}\Psi^{\delta}=(\tilde{\sigma}^{0}\sigma^{j})^{\delta}{}_{\alpha}\nabla_{0}X_{j}+C^{\delta}{}_{\alpha} \tag{2.43}$$

however the solution is not unique because we can add an antisymmetric matrix C. To go further we can calculate the matrix C from the covariant derivative algebra (1.11) acting on the X_i

$$\{\nabla_{\alpha},\nabla_{\beta}\}X_{i}=2i(\sigma^{0})_{\alpha\beta}\nabla_{0}X_{i}+[F_{\alpha\beta},X_{i}]$$

which is equal to

$$2i(\sigma^0)_{\alpha\beta}\nabla_0 X_i + [F_{\alpha\beta}, X_i] = i(\sigma_i)_{\beta\delta}\nabla_\alpha\Psi^\delta + i(\sigma_i)_{\alpha\delta}\nabla_\beta\Psi^\delta \ .$$

This equation has the solution

$$\nabla_\alpha\Psi^\delta = (\tilde{\sigma}^0\sigma^j)^\delta{}_\alpha\nabla_0 X_j + \frac{1}{4}(\tilde{\sigma}^j\sigma^k - \tilde{\sigma}^k\sigma^j)^\delta{}_\alpha[X_j, X_k]$$

which is consistent with the previous one (2.43) and moreover contains the antisymmetric matrix explicitly.

We can calculate the equation of motion for the X_i with the following algebraic (1.11) manipulation on intermediate results, *i.e.* the previous results is used in the

$$\{\nabla_\alpha, \nabla_\beta\}\Psi^\delta = 2i(\sigma^0)_{\alpha\beta}\nabla_0\Psi^\delta + [F_{\alpha\beta}, \Psi^\delta]$$

and this implies

$$\nabla_0\Psi^\delta - (\tilde{\sigma}^0\sigma^j)^\delta{}_\epsilon[\Psi^\epsilon, X_j] = 0 \qquad (2.44)$$

where we have also used

$$[\nabla_\beta, \nabla_0]X_i = -[F_{0\beta}, X_i]$$

for derivative commutation. Applying the ∇_α on the equation (2.44) gives us

$$\nabla_\alpha\nabla_0\Psi^\delta - (\tilde{\sigma}^0\sigma^j)^\delta{}_\epsilon\nabla_\alpha[\Psi^\epsilon, X_j] = 0 \qquad (2.45)$$

where we can again use a similar commutator

$$[\nabla_\beta, \nabla_0]\Psi^\delta = -\{F_{0\beta}, \Psi^\delta\} \ .$$

The equation of motion can be extracted from the equation (2.45). In fact we obtain two separate sets. The first set is obtained from the trace on fermionic indices and the second set comes from trace of the (2.45) multiplied by $(\tilde{\sigma}^0\sigma^l)^\beta{}_\delta$

$$\begin{aligned} 0 &= [\nabla_0 X^j, X_j] + \frac{i}{2}(\sigma_0)_{\alpha\beta}\{\Psi^\alpha, \Psi^\beta\} \\ 0 &= \nabla_0\nabla_0 X^l - [[X^l, X_i], X^i] - \frac{i}{2}(\sigma^l)_{\alpha\beta}\{\Psi^\alpha, \Psi^\beta\} \ . \end{aligned} \qquad (2.46)$$

Remember the fields X^l, Ψ^α are in the adjoint representation of the gauge group $SU(N)$. We show how these equations are related to an Yang-Mills theory now.

3 $D0$-branes Quantum Chemistry

3.1 Introduction

We provide here a short summary of the paper [9] in our current notation and see the correspondence to the result above. The low-energy limit of N $D0$-branes is described by a dimensional reduction of $\mathcal{N}=1$ supersymmetric Yang-Mills theory with gauge group $SU(N)$ in 1+9 dimension to 1+0 dimensional space [33]. We compare equations of motion comming from the reduced action and the result (2.46) of the previous section. They should be equivalent as this was the goal of the derivation in the target flat IIA superspace. A simplified system of two $D0$-branes is quantized and solved after this check.

The $D0$-branes has the gauge group $SU(N)$ with the generator basis T_ρ where we choose the Killing form to be identity matrix and the structure coefficinets are

$$[T_\rho, T_\sigma] = f_{\rho\sigma}{}^{\tau} T_\tau \ .$$

The reduced action contains fields X_i, Ψ^α, A_0 in the adjoint representation of the gauge group $SU(N)$. The $X_i = X_i{}^\rho T_\rho$ transforms in the vector representation of the $SO(9)$ group and the Ψ^α is Majorana-Weyl chiral spinor in 1+9 dimension. All these fields are only functions of one variable, let us use x^0. The action [9] reads

$$\begin{aligned} S &= \int \mathrm{d}x^0 \Big(\frac{1}{2g_s} \dot{X}_i{}^\rho \dot{X}_i{}^\rho + \frac{i}{2} \Psi^{\alpha\rho} \dot{\Psi}^{\alpha\rho} - \frac{1}{4g_s} \left(f^{\rho\sigma\tau} X_i{}^\sigma X_j{}^\tau \right)^2 + \frac{i}{2} f^{\rho\sigma\tau} X_i{}^\rho \Psi^{\alpha\sigma} (\sigma^i)_{\alpha\beta} \Psi^{\beta\tau} \\ &\quad + \frac{1}{g_s} f^{\rho\sigma\tau} \dot{X}_i{}^\rho A_0{}^\sigma X_i{}^\tau + \frac{1}{2g_s} \left(f^{\rho\sigma\tau} A_0{}^\sigma X_i{}^\tau \right)^2 - \frac{i}{2} f^{\rho\sigma\tau} A_0{}^\rho \Psi^{\alpha\sigma} \Psi^{\alpha\tau} \Big) \ . \end{aligned} \tag{3.47}$$

The field $A_0{}^\rho$ is a Lagrange multiplier and its variation generates the equation of motion in the gauge $A_0 = 0$

$$0 = \frac{1}{g_s} [\dot{X}^i, X_i]^\rho + \frac{i}{2} \{\Psi^\alpha, \Psi^\alpha\}^\rho \equiv G^\rho \tag{3.48}$$

which is a constraint in fact. The proper equations of motion generated by the variation of the field $X_i{}^\rho$ is in the $A_0 = 0$ gauge

$$0 = \frac{1}{g_s} \partial_0 \partial_0 X^{l\rho} - \frac{1}{g_s} [[X^l, X_i], X^i]^\rho - \frac{i}{2} (\sigma^l)_{\alpha\beta} \{\Psi^\alpha, \Psi^\beta\}^\rho \ .$$

We see that these two equations for $g_s = 1$ are exactly the same as the set (2.46). On quantum level the operator G^ρ defined in the constraint (3.48) restcricts our Hilbert space to vectors which satisfy

$$G^\rho \left| \Psi \right\rangle = 0 \ , \tag{3.49}$$

i.e. our physical space is gauge invariant because G^ρ are gauge generators.

3.2 Toy Model

We continue only with the simplified system of two $D0$-branes in three dimensional Minkowski space and hope that this gives us the basic behavior and also an understanding of the full problem. The Hamiltonian derived from the action (3.47) takes the form

$$\begin{aligned} H &= \frac{g_s}{2}\left({\pi_i}^\rho\right)^2 + \frac{1}{4g_s}\left(\epsilon^{\rho\sigma\tau}{X_1}^\sigma {X_2}^\tau\right)^2 \\ &\quad -\frac{1}{2}\epsilon^{\rho\sigma\tau}{X_1}^\rho\left(\chi^\sigma\chi^\tau - \bar{\chi}^\sigma\bar{\chi}^\tau\right) - \frac{i}{2}\epsilon^{\rho\sigma\tau}{X_2}^\rho\left(\chi^\sigma\chi^\tau + \bar{\chi}^\sigma\bar{\chi}^\tau\right) \end{aligned} \tag{3.50}$$

where X_i and the complex fermion χ are in the $SU(2)$ adjoint representation. Of course we have to impose gauge invariance (3.49). In fact, the gauge invariance complicates things somewhat since we would like to separate out gauge invariant degrees of freedom from pure gauge degrees of freedom in our quantum mechanical operators ${X_i}^\rho$ and ${\pi_i}^\rho$.

Let us focus on physical content of the ${X_i}^\rho$. It contains six components *(the gauge index ρ runs over three values and the space index $i = 1, 2$)*. We know that we can remove three of these variables using gauge transformations so only three variables are observable. These three variables should describe the relative position of two pointlike objects in two space dimensions. We draw the conclusion that one of the physical variables do not have the interpretation of a coordinate but rather as some internal auxilliary degree of freedom.

To get some further insight into this problem it is neccesery to investigate the bosonic vacuum of the theory. It is possible to explicitly separate the gauge degrees of freedom from ${X_i}^\rho$ by decomposition in matrix form [34]

$$(X)_{\rho i} = (\psi)_{\rho r}(\Lambda)_{rs}(\eta)_{si}\ . \tag{3.51}$$

Here the matrix ψ is an group element in the adjoint representation of $SU(2)$. Thus when the gauge group acts on ${X_i}^\rho$, ψ just changes by ordinary gauge group multiplication *(from the left)*. This decomposition has the advantage that all the gauge dependence sits in ψ and all the other matrices are gauge invariant.

In an analogous way we have separated out the dependence on rotations in space. Namely, performing an $SO(2)$ rotation in space we have an element of $SO(2)$ acting from the right on the matrix $X_{\rho i}$. Thus we can separate out the dependence on the angle in space *(we call the angle ϕ)* by saying that η is a group element of $SO(2)$.

We are left with the matrix Λ which by construction is both gauge and space rotation invariant

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ 0 & 0 \end{pmatrix} .$$

The bosonic potential in (3.50) is gauge and rotation invariant and in the new decomposition coordinates depends only on two λ_i which have length dimension (Fig. 1).

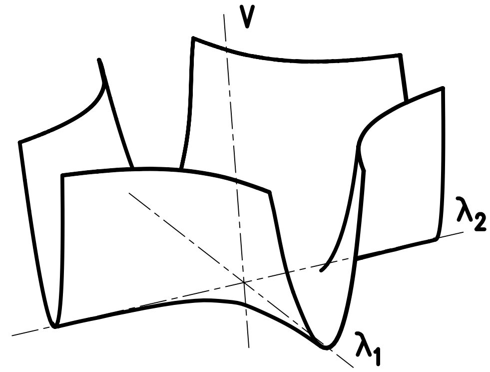

Figure 1: Bosonic potential

The parametrisation

$$\lambda_1 = r\cos\theta;\ \lambda_2 = r\sin\theta$$

is the simplest way how to obtain exactly one variable, r, with the dimension length, which could represent the relative distance of the two branes. The dimensionless θ is the auxilliary coordinate. The potential in this coordinate reads

$$\frac{1}{8g_s} r^4 \sin^2 2\theta\ .$$

Looking at the picture we can draw some interesting conclusions. If we fix a point on the bosonic vacuum *(a classical static configuration with minimum energy)*, that is on the axes, we can study the behavior of the potential for small fluctuations of the auxilliary variable θ. We see that for large r the θ fluctuation are very much suppressed but at small r, θ will be allowed to fluctuate. This can be interpreted to mean that when the branes get close to each other, they can start to move also in the θ direction. Thus, θ is an auxilliary coordinate which is visible only when the branes come close together.

The above discussion included only the bosonic degrees of freedom, we should keep in mind that the fermionic degrees of freedom can *(and will)* change this behavior somewhat. In essence, the Pauli repulsion will try to spread out the wavefunction as much as possible.

3.3 Ground State

To find the ground state we will find all gauge invariant states with spin zero. The first state is the vacuum state $|0\rangle$. Then we may act with the fermionic creation operators χ^ρ on the vacuum to find new states. The following states has total spin zero and they are gauge invariant

$$|r\rangle = \frac{1}{2}\psi_{\rho r}\,|\rho\rangle = \frac{1}{2}e^{i\phi}\psi_{\rho r}\epsilon^{\rho\sigma\tau}\chi^\sigma\chi^\tau\,|0\rangle\ . \tag{3.52}$$

That is, they satisfy

$$G^\rho\,|r\rangle = 0\ .$$

The most general gauge invariant wavefunction with total spin zero can then be written

$$g(r,\theta)\,|0\rangle + f_r(r,\theta)\,|r\rangle\ . \tag{3.53}$$

We define the wavefunction on the interval $\theta \in [0, \pi/4]$ because the map (3.51) is one to one here. It is not necessary to choose this particular interval, one could, for instance, select the interval $[-\pi/4, 0]$ instead of the above mentioned. Using the identification $\tilde\theta = -\theta$ the Hamiltonian defined on the interval $\tilde\theta \in [-\pi/4, 0]$ acting on the states with total spin zero is connected to our original Hamiltonian on the interval $\theta \in [0, \pi/4]$ by the unitary transformation

$$H(\tilde\theta) = U^\dagger H(\theta) U$$

where

$$U = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$

It is also possible to consider other interval $\tilde\theta \in [\pi/4, \pi/2]$. Using the identification $\tilde\theta = \pi/2 - \theta$ the corresponding Hamiltonian can be also obtained by a unitary transformation with the matrix

$$U = \begin{pmatrix} 1 & & & \\ & & -\mathrm{i} & \\ & \mathrm{i} & & \\ & & & 1 \end{pmatrix}.$$

The wavefunctions of course also transform under the unitary transformation

$$|\Psi(\theta)\rangle = U|\Psi(\tilde\theta\,)\rangle\ . \tag{3.54}$$

If we require that the wavefunctions be everywhere smooth, the above condition severely restricts the possible wavefunctions and in particular the basis wavefunctions that we can use. This principle was not used in [4] and hence the θ derivative of their groundstate wavefunction at $\theta = \pi/4$ is not well defined.

3.4 Numerical Method

We used [9] the numerical renormalized Numerov method [5] to solve the boundstate of the Hamiltonian (3.50). A brief description of its application together with results are presented here.

The Numerov method can solve discrete spectra of a one dimensional operator of the form

$$-\frac{1}{2}\frac{\partial^2}{\partial x^2}\mathbf{1} + \mathbf{V}(x) \tag{3.55}$$

which acts on $L[a,b] \otimes \mathbf{C}^n$ with Dirichlet boundary conditions. One cannot directly apply this method to solve the boundstate because the corresponding Hamiltonian is two dimensional as we want to solve the uknown functions (3.53) depending on two variables r and θ. The dependence on the other coordinates, the angles, has been already solved by the requirement that we are studying only gauge invariant states with spin zero.

Let us sketch shortly how to modify our problem to be able to use the method. An arbitrary wavefunction can be written in the form

$$|\Psi\rangle = \sum_{i,j} \Psi_{ij}(r) Y_{ij}(\theta) e_i \tag{3.56}$$

where $\{Y_{ij}(\theta), j = 1, \ldots\}$ is a complete basis of functions in θ which satisfy the consistency condition (3.54). To use this procedure on a computer we need to cut off the complete basis to have a finite number of basis vectors

$$Y_{ij} e_i \ \textit{(no sum)} \tag{3.57}$$

where their expectation values gives us coupled equations for the radial part $\Psi_{ij}(r)$. This practically leads to the desired one dimensional problem (3.55) *(please see the paper [9] for details)*.

It is not obvious how much our results for a fixed number of basis functions fit the exact solutions which one would get using the complete basis. To get some intuition for how the general solution would look like we will repeat the calculations increasing the number of basis functions each time and hopefully one can extrapolate the result to the exact case. At least we should be able to make an intelligent guess at the properties of the exact solution. The groundstate of our Hamiltonian is a good test for the method described above as it should have zero energy because of supersymmetry [35].

There are some results for the coupling constant $g_s = 0.1$ in the table 1 where N is the number of the test functions (3.57) and E is their corresponding groundstate energy. The columns N_p and E_p has the same meaning but they are related to the Wosiek method [6]. We see here that the boundstate energies E_p are quite comparable but the number of basis functions N_p *(which corresponds to our basis functions)* increases drastically.

The energy dependence on N is approximately

$$E = \frac{1.44}{N} \tag{3.58}$$

as was claimed in [6]. One can therefore predict the groundstate energy as a function of the number of basis functions with very high accuracy. We therefore see that in any concrete numerical calculation *(using a finite basis)* we do not expect to get zero energy.

N_p	E_p	N	E
4+2	0.395	1+1	0.376
10+8	0.303	2+2	0.295
20+20	0.216	2+3	0.214
1540+440		10+10	0.075
11480+3080		20+20	0.037
37820+9920		30+30	0.025
		40+40	0.018
		50+50	0.014
		80+80	0.008

Table 1: Results

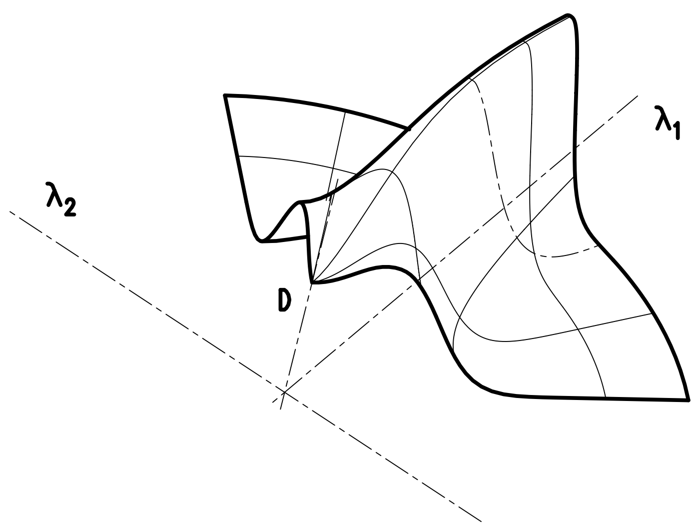

Figure 2: Probability density

The picture (Fig. 2) is the probability density for the case with the highest number of basis functions in the table. The domain of this plot is $(r, \theta) \in [0, 2.2] \times [-\pi/4, \pi/4]$. The hill of the probability density is located at the boson potential valley (3.52) and isolines represents sections for fixed r, θ and one for fixed density on each picture. The maximum of the probability density of any constant r section is in the potential valley ($\theta = 0$). Notice that the global maximum is not at $r = 0$. This is probably an effect of the fermion Pauli repulsion.

Increasing the number of basis functions, the only thing that happens is that the global maximum moves slowly to larger and larger r at the same time as the whole wavefunction becomes more spread out in r but more peaked in θ. One can assume that considering the complete base (3.57) the ground state density will be the same near the origin as in the picture and also will have the hills on the valleys of the potential which will be sharper and sharper when we follow the potential valley to large r. One can not conjecture in this case the behavior of the hills by this numerical method. Rather one has to use other methods [34] for the asymptotic behavior of the wavefunction at large r.

3.5 Summary

The purpose of the paper [9] was to find an approach how to calculate the boundstate of two $D0$-branes. This we have achieved with good results. We have also calculated the groundstate probability density near the origin with high accuracy which gives a basic intuition about the physics of branes on the string scale. In particular about the meaning of the auxilliary coordinates which become important at small distances between the branes. It is for example interesting to observe that the most probable position of the branes is not on top of each other but rather at some small distance away from each other. This we understand as an effect of the Pauli fermionic repulsion.

Appendix

Dirac Matrices

The Dirac matrices Γ^a for the $SO(1,9)$ group satisfy a Clifford algebra $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$ with the metric $\eta = diag(-1, 1, \ldots)$. We choose ten real

$$\Gamma^a = \begin{pmatrix} & (\gamma^a)^\alpha{}_{\dot\beta} \\ (\tilde\gamma^a)^{\dot\beta}{}_\alpha & \end{pmatrix} \tag{A.1}$$

which can act on the already mentioned (1.9) spinors $V^\alpha, V^{\dot\alpha}$. Numericaly, the matrices for $a = 0$ read $\gamma^0 = I = -\tilde\gamma^0$ where the I is the identity matrix and for the other ones are traceless symmetric matrices [36] such that $\gamma^a = \tilde\gamma^a$.

It is convenient to define Pauli matrices

$$\begin{pmatrix} (\sigma^a)_{\alpha\beta} & 0 \\ 0 & (\tilde\sigma^a)_{\dot\beta\dot\alpha} \end{pmatrix} = C \begin{pmatrix} 0 & (\gamma^a)^\alpha{}_{\dot\beta} \\ (\tilde\gamma^a)^{\dot\beta}{}_\alpha & 0 \end{pmatrix} \tag{A.2}$$

where the matrix C is the conjugation matrix

$$C = \begin{pmatrix} & c_{\alpha\dot\beta} \\ c_{\dot\beta\alpha} & \end{pmatrix} \equiv \begin{pmatrix} & I \\ -I & \end{pmatrix} .$$

The Pauli matrices satisfy

$$(\sigma^a)_{\alpha\beta}(\tilde\sigma^a)^{\beta\gamma} + (\sigma^b)_{\alpha\beta}(\tilde\sigma)^{\beta\gamma} = -2\eta^{ab}\delta_\alpha{}^\gamma \tag{A.3}$$

together with Fierz identity

$$(\sigma^a)_{(\alpha\beta}(\sigma_a)_{\gamma)\delta} = 0 . \tag{A.4}$$

Transformation

We start with transformation properties of the tangent vector V^A (1.9) under the Lorentz group together with spinor notation. Then the covariant derivative (1.8) is discussed.

We have already mentioned the V^a transforms in the vector representation of the Lorentz group $SO(1,9)$ and the fermionic part V^α is the chiral spinor in the **16** and $V^{\dot\alpha}$ is antichiral spinor in the $\underline{\mathbf{16}}$.

The matrix Λ which transforms the vector V^a and the matrix S which transforms both spinors $V^\alpha, V^{\dot\alpha}$ are related with each other [37]

$$\Lambda^b{}_a\Gamma^a = S^{-1}\Gamma^b S$$

where the Γ^a are the Dirac matrices. This relation for the infinitesimal transformations $\delta\Lambda \equiv 1/2\delta\omega_{cd}M^{dc}, \delta S = 1/2\delta\omega_{cd}L^{dc}$ implies

$$(M^{ab})^c{}_d\Gamma^d = [\Gamma^c, L^{ab}] . \tag{A.5}$$

The solution of this equation are generators

$$(M^{dc})^a{}_b = \eta^{da}\delta^c{}_d - \eta^{ca}\delta^d{}_b \,, \qquad L^{dc} = \frac{1}{4}[\Gamma^d, \Gamma^c] \,. \tag{A.6}$$

The matrix L^{dc} is block diagonal

$$L \equiv \begin{pmatrix} L^+ & 0 \\ 0 & L^- \end{pmatrix}$$

for our Γ^a choice and moreover $L^{+,T} = -L^-$. We see that the spinors V^α has the transformation matrix L^+ and $V^{\dot\alpha}$ L^-. There are also spinors $V_\alpha, V_{\dot\beta}$ defined with help of the conjugation matrix

$$\begin{pmatrix} V_\alpha \\ V_{\dot\beta} \end{pmatrix} \equiv \begin{pmatrix} & c_{\alpha\dot\delta} \\ c_{\dot\beta\gamma} & \end{pmatrix} \begin{pmatrix} V^\gamma \\ V^{\dot\delta} \end{pmatrix} \,.$$

The spinor V_α has the transformation matrix numerically equal to L^+ and $V_{\dot\alpha}$ to L^-. This is the reason why we can lower or raise indices in the manner $V_\alpha \equiv V^{\dot\alpha}$ and $V_{\dot\alpha} \equiv V^\alpha$.

Covariant Derivative

We are ready to write the covariant derivative (1.8) more explicitly. For example

$$\nabla V^a = dV^a + \frac{1}{2} V^b \wedge \omega_{cd} \left(M^{dc}\right)^a{}_b = dV^a + V^b \wedge \omega_b{}^a \tag{A.7}$$

where the M^{dc} is the group generator from (A.6). This corresponds to our initial covariant derivative definition (1.1). We have

$$\nabla V^\beta = dV^\beta + \frac{1}{2} V^\alpha \wedge \omega_{cd} \left(L^{+dc}\right)^\beta{}_\alpha \tag{A.8}$$

for the chiral spinor. Because of the identification $V_\alpha \equiv V^{\dot\alpha}$ it also holds $\nabla V_\alpha = \nabla V^{\dot\alpha}$

$$\begin{aligned} \nabla V^{\dot\beta} &= dV^{\dot\beta} + \frac{1}{2} V^{\dot\alpha} \wedge \omega_{cd} \left(L^{-dc}\right)^{\dot\beta}{}_{\dot\alpha} \\ &= dV_\beta + \frac{1}{2} V_\alpha \wedge \omega_{cd} \left(L^{-dc}\right)_\beta{}^\alpha = \nabla V_\beta \,. \end{aligned}$$

Let us calculate the covariant derivative of the torsion component (1.14)

$$\begin{aligned} \nabla T_{\beta\gamma}{}^b &= -i\nabla(\sigma^b)_{\beta\gamma} \\ &= -\frac{i}{2}\omega_{cd}(M^{dc})^b{}_a(\sigma^a)_{\beta\gamma} - \frac{i}{2}\omega_{cd}(L^{-dc})_\beta{}^\epsilon(\sigma^b)_{\epsilon\gamma} - \frac{i}{2}\omega_{cd}(L^{-dc})_\gamma{}^\epsilon(\sigma^b)_{\beta\epsilon} \\ &= -\frac{i}{2}\omega_{cd}(M^{dc})^b{}_a(\sigma^a)_{\beta\gamma} + \frac{i}{2}\omega_{cd}(\sigma^b)_{\epsilon\gamma}(L^{+dc})^\epsilon{}_\beta - \frac{i}{2}\omega_{cd}(L^{-dc})_\gamma{}^\epsilon(\sigma^b)_{\beta\epsilon} = 0 \end{aligned} \tag{A.9}$$

because the last line is numerically equivalent to the relation (A.5). We can use this result to proove the equation (2.34)

$$Dh_\beta{}^{\dot\alpha} = -(\sigma^b)_\beta{}^{\dot\alpha} u_b{}^i \hat\Omega_i{}^0 \ .$$

The $Dh_\beta{}^{\dot\alpha}$ is equal to the pullback of the form $\nabla h_\beta{}^{\dot\alpha}$ where the ∇ is the target superspace covariant derivative. The form is

$$\nabla h_\beta{}^{\dot\alpha} = \nabla h_{\beta\alpha} = \nabla(\sigma^b)_{\beta\alpha} u_b{}^0 + (\sigma^b)_{\beta\alpha} \nabla u_b{}^0 = -(\sigma^b)_{\beta\alpha} u_b{}^i \Omega_i{}^0 \ . \tag{A.10}$$

where we used the fact $\nabla(\sigma^b)_{\beta\alpha} = 0$ and the definition (2.22). The pullback of equation (A.10) completes the proof.

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