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Supersymmetric Gauge Theory and String Theory

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Dissertation Abstract:
Supersymmetry is a powerful symmetry that imposes severe restrictions on the theory. One of the consequences is the restriction on the effective action. For $\mathcal{N} = 2$ supersymmetric Yang-Mills theory the effective action is given by a single holomorphic function. This fact together with electromagnetic duality can be combined to find the explicit form of the effective action. The effective action is described in terms of an elliptic curve and suitable period integrals, which describe the scalar field and its dual field. All of these objects depend on one or several moduli - gauge invariant parameters that describe the theory.
The description of the effective action uses extensively magnetic monopoles. The Bogomol'nyi-Prasad-Sommerfield monopoles can be used to combine the relationship between the period integrals and moduli on one hand, and the spatial dependence of the scalar field on the other to find the spatial dependence of the moduli themselves. In this text the spatial dependence of the moduli is studied for $\mathcal{N} = 2$ super-Yang-Mills theories with gauge groups SU(2) and SU(3). To begin with, a first order differential equation for the modulus of SU(2) is derived and the solutions are studied both numerically and analytically. The solutions can be divided into several groups according to their behavior. The different types of behavior also affect the magnetic and electric field as well as the energy density function.
For SU(3) it is necessary to generalize the procedure used for SU(2). After that a system of differential equations is derived for the moduli. The moduli space is larger than for SU(2), and more difficult to visualize. Suitable examples of curves of marginal stability were chosen which are both representative and can be presented also in three dimensional space. Suitable numerical solutions to the differential equations which can be easily visualized are presented as well.

Abstrakt:
Supersymetrie je symetrie, která zcela zásadně zužuje možnosti teorie. Jedním z důsledků této skutečnosti jsou požadavky na efektivní akci. Pro $\mathcal{N} = 2$ supersymetrickou Yangovu-Millsovu teorii je efektivní akce dána jedinou holomorfní funkcí, což spolu s elektromagnetickou dualitou umožňuje najít její explicitní tvar. Efektivní akce je popsána pomocí eliptické křivky a vhodných integrálů - tzv. period, které popisují skalární pole a duální skalární pole. Všechny tyto objekty závisí na jednom či více kalibračně invariantních parametrech, jež popisují teorii, tzv. modulech. Popis efektivní akce je založen na magnetických monopolech. Pomocí Bogomolnyiho-Prasadova-Sommerfieldova monopolu lze zkombinovat vztah mezi periodami a moduly na jedné straně a mezi skalárním polem a souřadnicemi v prostoru na straně druhé. V této práci je studována prostorová závislost modulů pro $\mathcal{N} = 2$ supersymetrickou Yangovu-Millsovu teorii s kalibračními grupami SU(2) a SU(3). Nejdříve je odvozena diferenciální rovnice prvního řádu pro SU(2) a vlastnosti jejich řešení jsou zkoumány numericky i analyticky. Řešení lze podle chování rozdělit do několika skupin. Toto rozdělení se vztahuje i na chování elektrických a magnetických polí a na hustotu energie. Pro SU(3) je nutno nejdříve zobecnit postup použitý pro SU(2). Poté jsou odvozeny diferenciální rovnice pro moduly. Prostor modulů je větší než v případě SU(2) a je obtížnější jej znázornit. Byly vybrány vhodné příklady křivek marginální stability, které jsou reprezentativní a lze je vizualizovat v trojrozměrném prostoru. Dále jsou ukázány vhodné příklady řešení diferenciálních rovnic, jež lze rovněž dobře znázornit.

Contents

Chapter 1

Introduction

The standard model of particle physics is a theory describing subatomic particles and their strong, weak and electromagnetric interactions and has been very successful at explaining a wide variety of experimental results. Calculations in the standard model are mostly based on perturbation theory. Especially in the theory of the strong interaction, Quantum Chromodynamics, computations at low energies, which correspond to observable physics, are very difficult, since perturbation theory is not applicable any more. In order to learn more about perturbative effects people have turned to other quantum field theories in which the perturbative effects are easier to describe. These theories serve as toy models, and hope is that the results can be transferred back to Quantum Chromodynamics. One of the most popular toy models are supersymmetric theories, especially $\mathcal{N} = 2$ supersymmetric Yang-Mills theory.

Any symmetry imposed on a theory reduces significantly the possible configuration choices: functions that describe particles or fields must transform in representations of the symmetry group, field equations must be invariant under this symmetry and it can happen that some effects are simply forbidden because they would violate the symmetry. Symmetries may involve internal degrees of freedom, e.g. charge, or they may involve symmetries of the spacetime itself.
The most common spacetime symmetry is the Poincaré symmetry of special relativity and the internal symmetries are mostly matrix groups of low rank. Effort has been made to combine these two types of symmetry (internal and spacetime) in a nontrivial way, i.e. to create an algebra which would include the ordinary spacetime algebra and in which the spacetime algebra would have nontrivial relations with the rest of the algebra (or at least part of it). However, Coleman and Mandula [1] showed that this is not possible within an ordinary Lie algebra. The way out, as shown by Haag, Lopuszanski and Sohnius [2] is to use a graded Lie algebra which is given by anticommutators as well as commutators. The corresponding, so-called fermionic, generators transform in a nontrivial way under the Lorentz group, creating in this way an extension of ordinary spacetime symmetry which is called supersymmetry. Supersymmetry is a fairly strict requirement: it restricts the matter content of theories, introduces specific internal symmetries and has also impact on the (non)perturbative behavior. Fields must form representations of the supersymmetry group which restricts our choice in terms of the spacetime fields. Superfields are formed by combinations of several spacetime fields, so that we always have com-

binations of bosonic and fermionic fields. In fact, it can be proven that there must be the same number of bosonic and fermionic degrees of freedom. There are further authomorphisms of the algebra, called R-symmetries which transform the fields in a nontrivial way. These can be exact quantum symmetries or broken by anomalies. In supersymmetric theories quantum corrections are suppressed. This is due to the fact that bosons and fermions contribute with opposite sign, which makes the corrections less severe. There are several varieties of supersymmetry, which have a different number of fermionic generators. The $\mathcal{N} = 2$ supersymmetric theory is especially popular, since it benefits from the restrictions that the two sets of fermionic generators impose (e.g. the effective action of the gauge field is given by a single holomorphic function and there are only one-loop corrections to the effective action) but still keeps enough of its nontrivial quantum behaviour to be interesting (unlike e.g. $\mathcal{N} = 4$ which has a trivial beta function).

Electromagnetic duality is a very old idea. In fact, already the form of the Maxwell equations raises the question why electric and magnetic degrees of freedom are treated so differently. Dirac [3] found a solution with a magnetic charge - a magnetic monopole, and showed that the magnetic charge must be quantized. A nonsingular solution was found by [4]. The quantization condition was later modified by Zwanziger and Schwinger to accomodate also dyons - particles with both electric and magnetic charges [5–7]. Solitons with a magnetic charge can be constructed in Yang-Mills-Higgs theories, e.g. the t'Hooft Polyakov monopole [8]. Montonen and Olive showed how exact electromagnetic duality could be implemented in various theories [9]. The concept of electromagnetic duality was also used in strongly coupled gauge theories. Here, the idea of having a dual formulation in terms of weakly coupled magnetic monopoles has proven itself very useful. Furthermore, the dynamics in these cases is closely related to properties of magnetic monopoles, that can be often studied semi-classically.

Seiberg and Witten used electromagnetic duality and especially the ideas of dual formulations in $\mathcal{N} = 2$ super Yang-Mills theories with gauge group SU(2) to determine the exact form of the effective action [10,11]. This approach was soon extended to other gauge groups and to the relation between moduli and spatial dependence of the monopole within this context.

The goal of this theses was to study the details of the spatial dependence of the monopole and related matters in SU(2) and to translate as much as possible to SU(3). The SU(3) monopoles that were used are embedded SU(2) monopoles, so we can expect similarities. On the other hand SU(3) possesses two moduli and thus we can expect some new behavior related to the moduli which has no SU(2) analogue. The details of the similarities resp. differences between the two cases were studied as well.

The thesis was written with the hope that some day somebody might read it and find it useful. Therefore the text is not as brief as it could be otherwise. Details have been filled in for various calculations, so that the steps between are "overcomable". This holds also for those cases, where results can be found in literature and their derivation is described only very briefly (and is not straightforward).
The thesis is organized as follows:
Chapter 2 contains a brief description of supersymmetry, since this is the key porop-

erty that allowed Seiberg and Witten to find an exact solution for the effective action in supersymmetric theories. Supersymmetry is a vast topic, it is clearly impossible to explain even only all those aspects that are used in the later parts so my aim was to give at least a first idea of its concepts and explain its role in the following.
In chapter 3 a general overview of monopole solutions in Yang-Mills-Higgs theories and supersymmetric Yang-Mills theories is presented. Special solutions as BPS monopoles and t'Hooft-Polyakov monopole are described.
In chapter 4 the Seiberg-Witten approach is reviewed in detail for SU(2) and generalizations to other gauge groups are given as well. Quantum corrections to BPS solutions are reviewed.
Chapters 5 and 6 are the core of this work. In these two chapters original results concerning quantum corrected monopoles for SU(2) and SU(3) theories are presented. Chapter 6 contains also definitions and generalizations of some aspects which are either trivial or well-known for SU(2).
Calculations which would disrupt the flow of the text were moved to the appendices. Some basic facts about the hypergeometric function and Appell's function are reviewed in appendix A. Appendix B contains expansions which are needed for the calculations in chapter 5. A few basic facts about SU(3) as well as notations and conventions used in this text are given in appendix C. Appendix D reviews a systematic way to set up Picard-Fuchs equations and for the two gauge groups SU(2) and SU(3) is described in more detail. Appendix E contains the derivation of the solution of the Picard-Fuchs equations for SU(3). More details were added to the results found in literature, so as to make each single step more clear.

Chapter 2

Supersymmetry

2.1 Review of Supersymmetry

In this section we shall give only the some basic facts about supersymmetry. We will restrict ourselves to facts, that will be useful in the following. For good introductions on supersymmetry see eg. [12], [13], [14] and [15], In the following we will use the notation and conventions of [15].

2.1.1 Superalgebras

Coleman and Mandula proved in [1], that under certain physically reasonable assumptions, the only possible symmetries of the S-matrix are :

- Poincaré invariance, which is the semi-direct product of translations and Lorentz rotations
- internal symmetries, forming a Lie algebra
- discrete symmetries C, P and T.

Additional symmetries can be realized if one uses graded Lie algebras instead of Lie algebras. Graded Lie algebras are generated not only by commuting generators (as ordinary Lie algebras) but also by anti-commuting generators. These "fermionic" generators transform as spinors under the Lorentz group, thus "enlarging" the space-time symmetry. The only graded Lie algebra consistent with relativistic field theory, is supersymmetry. Supersymmetry in four dimensions, as proved by Haag, Lopuszanski and Sohnius in [2], includes the bosonic generators of the Poincaré group $(J_{\alpha\beta}, \bar{J}_{\dot{\alpha}\dot{\beta}}, P_{\alpha\dot{\beta}})$, $\mathcal{N}$ fermionic generators $Q_{a\alpha}$ and their conjugates $\bar{Q}^a_{\dot{\alpha}}$ and at most $\frac{1}{2}\mathcal{N}(\mathcal{N}-1)$ complex central charges $Z_{ab} = -Z_{ba}$. The central charges are called central because they commute with all generators in the algebra. Greek indices refer to the spinor representation of the Lorentz group. Since all irreducible representations can be classified by two halfintegers (a, b), any quantity can be represented by a quantity with $2a$ undotted and $2b$ dotted indices. The spinor representation $(\frac{1}{2}, 0)$ is labeled by Greek indices ψ^α, the conjugate representation $(0, \frac{1}{2})$ by dotted

Greek indices $\bar{\psi}^{\dot{\alpha}}$, the vector representation $(\frac{1}{2},\frac{1}{2})$ by one dotted and one undotted index $V^{\alpha\dot{\beta}}$. We shall denote the combination of a dotted and undotted Greek index by an underlined Latin index $\underline{a} \equiv (\alpha\dot{\alpha})$. The self-dual and anti-self-dual parts of an antisymmetric tensor have two undotted, respectively, dotted indices. The Latin index a runs from 1 to $\mathcal{N}$. Brackets denote symmetrization without any factors. The algebra is

$$\begin{aligned}
\{Q_{a\alpha}, \bar{Q}^b_{\dot{\beta}}\} &= \delta^b_a P_{\alpha\dot{\beta}} \\
\{Q_{a\alpha}, Q_{b\beta}\} &= C_{\alpha\beta} Z_{ab} \\
[Q_{a\alpha}, P_{\beta\dot{\beta}}] &= [P_{\alpha\dot{\alpha}}, P_{\beta\dot{\beta}}] = [\bar{J}_{\dot{\alpha}\dot{\beta}}, Q_{c\gamma}] = 0 \\
[J_{\alpha\beta}, Q_{c\gamma}] &= \frac{1}{2} i C_{\gamma(\alpha} Q_{c\beta)} \\
[J_{\alpha\beta}, P_{\gamma\dot{\gamma}}] &= \frac{1}{2} i C_{\gamma(\alpha} P_{\beta)\dot{\gamma}} \\
[J_{\alpha\beta}, J^{\gamma\delta}] &= -\frac{1}{2} i \delta^{(\gamma}_{(a} J^{\delta)}_{\beta)} \\
[J_{\alpha\beta}, \bar{J}_{\dot{\alpha}\dot{\beta}}] &= [Z_{ab}, Z_{cd}] = [Z_{ab}, \bar{Z}^{cd}] = 0.
\end{aligned}$$

Here $C_{\alpha\beta}$ is the Pauli matrix σ_2. An internal symmetry can be added by adding generators of the corresponding Lie algebra T_A (A is the group index) and the commutation relations

$$\begin{aligned}
[T_A, T_B] &= i f^C_{AB} T_C \\
[T_A, P_{\alpha\dot{\alpha}}] &= [T_A, J_{\alpha\beta}] = [T_A, \bar{J}_{\dot{\alpha}\dot{\beta}}] = 0 \\
[T_A, Z_{ab}] &= 0 \\
[T_A, Q_{a\alpha}] &= -t_{Aab} Q_{b\alpha}
\end{aligned}$$

where f^C_{AB} are the structure constants of the Lie algebra and the coefficients $t_{A\alpha\beta}$ furnish a representation

$$[t_A, t_B] = i f^C_{AB} t_C.$$

An automorphism of the algebra which mixes the various generators $Q_{a\alpha}$ is called R-symmetry.

2.1.2 Particle Representations

The particle content of supersymmetric representations can be analyzed in terms of representations of the Poincaré group. As in the Poincaré group, P^2 is a Casimir (it commutes with all the generators). Therefore, all representations can be characterized by their mass. This implies further, that all states in a given irreducible representation have the same mass.

Massive states, no central charge

In the rest frame the momentum operator can be written as

$$P_{\alpha\dot{\alpha}} = \begin{pmatrix} P_0 + P_3 & P_1 - iP_2 \\ P_1 + iP_2 & P_0 - P_3 \end{pmatrix},$$

in the rest frame this means that it is proportional to the unit matrix $P_{\alpha\dot{\alpha}} = -m\delta_{\alpha\dot{\alpha}}$. The fermionic generators act as creation/ annihilation operators (up to rescaling) [1]

$$a_{a\alpha} = \frac{i}{\sqrt{m}} Q_{a\alpha} \qquad\qquad (a_{a\alpha})^\dagger = \frac{i}{\sqrt{m}} \bar{Q}_{a\dot{\alpha}}.$$

The particle content can be constructed as usual from the Clifford vacuum. The Clifford vacuum forms an irreducible representation of the Poincaré group and thus has definite spin j. We have $2\mathcal{N}$ anticommuting creation operators, which since they are in the representation $(0, \frac{1}{2})$, raise, resp., lower the spin by $\frac{1}{2}$. The total number of states is $2^{2\mathcal{N}}k$, where k is the number of states in the Clifford vacuum.

Massless states, no central charges

The construction is analogous to the massive case. In a suitable Lorentz frame with only $p_{+\dot{+}} = P_0 + P_3$ nonzero, we find that also $\{Q_{a-}, \bar{Q}^b_{\dot{-}}\} = 0$. If the anticommutator of an operator with its conjugate vanishes, the operator itself must vanish identically when acting on any states. Therefore we are left with only half of the creation/annihilation operators of the massless case, those related to Q_{a+}. These lower the helicity by $\frac{1}{2}$, so that if the helicity of the Clifford vacuum is λ, the helicities of the states are $\lambda, \cdots, \lambda - \mathcal{N}\frac{1}{2}$. The total number of states is $2^{\mathcal{N}}k$, where k is the number of states in the Clifford vacuum.

Nonzero central charges

If there are nonzero central charges, it is convenient to change the basis, so that they can be written as

$$Z_{ab} = \begin{pmatrix} Z_{1ij} & 0 & \dots & 0 \\ 0 & Z_{2ij} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Z_{\frac{\mathcal{N}}{2}ij} \end{pmatrix}$$

where Z_{Mij} is an antisymmetric 2×2 matrix $Z_{Mij} = Z_M \epsilon^{ij}$, $\epsilon^{ij} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. If $\mathcal{N}$ is odd there is an extra column and row of zeroes. The Z_M, $M = 1, \dots, \mathcal{N}/2$ are real and nonnegative. After changing the index a, $a = 1, \dots, \mathcal{N}$ for the indices (Mi), $M = 1, \dots, \mathcal{N}/2$, $i = 1, 2$, the anticommutation relations of the fermionic generators become (in the rest frame),

$$\begin{aligned} \{Q^{Li}_\alpha, \bar{Q}^{Mj}_{\dot{\beta}}\} &= -m\delta^{ij}\delta^{LM}\sigma^0_{\alpha\dot{\beta}} \\ \{Q^{Mi}_\alpha, Q^{Lj}_\beta\} &= C_{\alpha\beta}\epsilon^{ij}\delta^{ML}Z_M \\ \{\bar{Q}^{Mi}_{\dot{\alpha}}, \bar{Q}^{Lj}_{\dot{\beta}}\} &= -C_{\dot{\alpha}\dot{\beta}}\epsilon^{ij}\delta^{ML}Z_M \end{aligned}$$

[1] Due to conventions in superspace we have an extra sign for hermitian conjugation $(\psi_\alpha)^\dagger = -\bar{\psi}_{\dot{\alpha}}$

We can construct $2\mathcal{N}$ creation/annihilation operators (σ^0 is the identity matrix)

$$\begin{aligned} a_\alpha^M &= \frac{1}{\sqrt{2}}\left(Q_{\alpha M1} - C_{\alpha\beta}\sigma^{0\beta\dot{\gamma}}\bar{Q}_{\dot{\gamma}M2}\right) \\ \left(a_\alpha^M\right)^\dagger &= \frac{1}{\sqrt{2}}\left(-\bar{Q}_{\dot{\alpha}M1} - C_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{0\dot{\beta}\gamma}Q_{\gamma M2}\right) \\ b_\alpha^M &= \frac{1}{\sqrt{2}}\left(Q_{\alpha M1} + C_{\alpha\beta}\sigma^{0\beta\dot{\gamma}}\bar{Q}_{\dot{\gamma}M2}\right) \\ \left(b_\alpha^M\right)^\dagger &= \frac{1}{\sqrt{2}}\left(-\bar{Q}_{\dot{\alpha}M1} + C_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{0\dot{\beta}\gamma}Q_{\gamma M2}\right), \end{aligned}$$

with nonzero anticommutators

$$\begin{aligned} \{a_\alpha^L, \left(a_\beta^M\right)^\dagger\} &= \delta^{LM}\delta_{\alpha\beta}(m + Z_M) \\ \{b_\alpha^L, \left(b_\beta^M\right)^\dagger\} &= \delta^{LM}\delta_{\alpha\beta}(m - Z_M). \end{aligned}$$

The anticommutator of an operator with its conjugate must be nonnegative, so since the Z_M's are nonnegative we get a lower bound on the masses

$$m \geq Z_M. \tag{2.1}$$

If $Z_M < m$ for all M we have a set of $2\mathcal{N}$ creation/annihilation operators, and the particle content is the same as if there were no central charges. If the bound is saturated for some Z_M, i.e. $Z_M = m$, the corresponding operators b_α anticommute with their adjoints and must be identically zero (this is analogous to the massless case). If $\mathcal{K}$ central charges saturate the bound there are $2(\mathcal{N} - \mathcal{K}) + \mathcal{K}$ creation/annihilation operators in the theory. States for which the bound is saturated are called BPS states in analogy to BPS monopoles in gauge theories.

2.1.3 Superfield Representations

Field representations can be conveniently described as fields in superspace, an extension of spacetime. We shall show here only the basic ideas for $\mathcal{N} = 1$.
Spacetime can be viewed as the coset space (Poincaré group) mod (Lorentz group), with the points in spacetime identified with the orbits that the Lorentz group sweeps out in the Poincaré group. In a similar way global flat superspace can be defined as (super-Poincaré group) mod (Lorentz group). Here it is necessary to add fermionic coordinates to the bosonic coordinates x, these fermionic coordinates are taken to be the anticommuting Grassmann variables $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$. An element of superspace can be parametrised using the exponential mapping

$$g(x, \theta, \bar{\theta}) = \mathrm{e}^{i(x^{\alpha\dot{\beta}}P_{\alpha\dot{\beta}} + \theta^\alpha Q_\alpha + \bar{\theta}^{\dot{\alpha}}, \bar{Q}_{\dot{\alpha}})}.$$

where P and Q's are abstract generators. The group structure tells us how the generators act and allows us to represent these generators in terms of differential operators.

$$\begin{aligned} \Phi(x, \theta, \bar{\theta}) = f(x) + \theta^\alpha\varphi_\alpha(x) + \bar{\theta}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}}(x) + \theta^2 m(x) + \bar{\theta}^2 n(x) + \\ + \theta_\alpha v^{\alpha\dot{\beta}}(x)\bar{\theta}_{\dot{\beta}} + \theta^2\bar{\theta}^{\dot{\beta}}\bar{\lambda}(x)_{\dot{\beta}} + \bar{\theta}^2\theta^\beta\chi(x)_\beta + \theta^2\bar{\theta}^2 d(x). \end{aligned}$$

From this we see that it contains 4 scalar fields $f(x)$, $m(x)$, $n(x)$, $d(x)$, 4 spinor fields $\varphi(x)$, $\bar{\psi}(x)$, $\bar{\lambda}(x)$, $\chi(x)$ and a vector field $v^{\alpha\dot{\beta}}$. Alternative to Taylor expansions are projections. We can define the components as the projections of the field and its spinor derivatives on the subspace $\theta = \bar{\theta} = 0$. This is denoted by $\Phi|$. Thus for the previous example we have

$$\begin{aligned} f(x) &= \Phi| \\ \varphi_\alpha(x) &= D_\alpha \Phi| \\ m(x) &= D^2\Phi\big|\,, \qquad \text{etc..} \end{aligned}$$

However representations in terms of general superfields contain more fields than we are interested in. By making a specific choice, e.g. by imposing constraints, we can find superfields that contain physical fields that we are interested in as i.e. the scalar field, gauge field, spinors.

Chiral superfield

Chiral superfields can be obtained by imposing $\bar{D}_{\dot{\alpha}}\Phi = 0$. It contains two scalar fields and a spinor field

$$\begin{aligned} \phi(x) &= \Phi| \\ \psi_\alpha(x) &= D_\alpha \Phi| \\ F(x) &= D^2\Phi\big|\,. \end{aligned} \tag{2.2}$$

Vector superfield

A vector superfield is a real scalar superfield $V = \bar{V}$. It can be shown that there exists a gauge, the Wess-Zumino gauge, in which only certain components are nonzero . In the Abelian case these are

$$\begin{aligned} \lambda_\alpha &= D_\alpha V| \\ A_{\alpha\dot{\alpha}} &= \frac{1}{2}[\bar{D}_{\dot{\alpha}}, D_\alpha]V\Big| \\ D' &= \frac{1}{2}D^\alpha \bar{D}^2 D_\alpha V\Big|\,. \end{aligned} \tag{2.3}$$

The vector field $A_{\alpha\dot{\alpha}}$ is the gauge field, the field λ_α is a spinor and D' is a scalar field.

Field strength multiplet

The field strength multiplet is a spinor superfield W_α which satisfies $D^\alpha W_\alpha = -\bar{D}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}}$. It contains a spinor field λ_α, a scalar D' and a symmetric bispinor $f_{\alpha\beta}$:

$$\begin{aligned} \lambda_\alpha &= W_\alpha| \\ f_{\alpha\beta} &= \frac{1}{2}D_{(\alpha}W_{\beta)}\Big| \\ D' &= -\frac{1}{2}iD^\alpha W_\alpha\Big|. \end{aligned} \tag{2.4}$$

In the Abelian case it can be expressed through the vector field as $W_\alpha = i\bar{D}^2 D_\alpha V$. Then the $f_{\alpha\beta}$ and its conjugate $f_{\dot{\alpha}\dot{\beta}}$ corrrespond to the self-dual and anti-self-dual parts of the field strength of the gauge field $v^{\alpha\dot{\beta}}$. This superfield is also called field strength multiplet.

Non-Abelian gauge group

If the gauge group is non-Abelian it is more convenient to work in terms of the gauge covariant derivatives $\nabla_A = D_A - i\Gamma_A$ with $A = \alpha, \dot{\alpha}, (\alpha\dot{\alpha})$. The field strengths F_{AB} are given by the (anti)-commutator of covariant derivatives

$$[\nabla_A, \nabla_B\} = T_{AB}^C \nabla_C - iF_{AB}, \tag{2.5}$$

where T_{AB}^C is the torsion and [} denotes either commutator or anticommutator. The spinor superfield is part of the fieldstrength

$$F_{\dot{\alpha},\beta\dot{\beta}} = iC_{\dot{\alpha}\dot{\beta}}W_\beta.$$

It is convenient to define the physical components of the chiral and spinor superfields as in the Abelian case. They will take the same form case upon changing covariant derivatives D, ∂ for gauge covariant derivatives ∇ in (2.3).

$\mathcal{N} = 2$ representations

$\mathcal{N} = 2$ representations can be found as superfields on $\mathcal{N} = 2$ superspace, with two pairs of anticommuting coordinates θ^1,θ^2,$\tilde{\theta}^1$, $\tilde{\theta}^2$. The vector multiplet can be decomposed with respect to $\tilde{\theta}$ into two $\mathcal{N} = 1$ multiplets: a scalar multiplet and a spinor multiplet

$$\begin{aligned} \Phi &= W|_{\tilde{\theta}=0} \\ W_\alpha &= -\tilde{\nabla}_\alpha W\Big|_{\tilde{\theta}=0}. \end{aligned} \tag{2.6}$$

The minus sign on the second line is a matter of convention. It follows that the matter field in Φ must be in the same representation of the gauge group as the gauge fields, i.e. in the adjoint representation.

2.2 $\mathcal{N} = 2$ Super Yang-Mills theory

2.2.1 $\mathcal{N} = 2$ Algebra

The $\mathcal{N} = 2$ algebra has two pairs of fermionic superspace coordinates. The commutations relations of the covariant derivatives are

$$\begin{aligned} \left\{\nabla_\alpha, \bar{\nabla}_{\dot\alpha}\right\} &= i\nabla_{\alpha\dot\alpha} & \{\nabla_\alpha, Q_\beta\} &= iC_{\alpha\beta}\bar{W} \\ \{Q_\alpha, Q_{\dot\alpha}\} &= i\nabla_{\alpha\dot\alpha} & \left\{\bar{\nabla}_{\dot\alpha}, \bar{Q}_{\dot\beta}\right\} &= iC_{\dot\alpha\dot\beta}W, \end{aligned} \tag{2.7}$$

all the other (anti)commutators are zero. Here we denote as Q_a the covariant derivative with respect to $\tilde\theta$, $Q_a = -i\tilde\nabla_\alpha$, in order to avoid too many ∇'s. The superfield W is the central charge of the algebra Z, $W = -iZ_{12} = -iZ$. We can get more relations by using the Jacobi identity, e.g. the following relation for the superfield W

$$Q^2W = \bar\nabla^2\bar{W}.$$

Thus W depends only on half of the superspace and by analogy with $\mathcal{N} = 1$ is called $\mathcal{N} = 2$ chiral.The $\mathcal{N} = 2$ chiral superfield is a vector multiplet and can thus be written in terms of an $\mathcal{N} = 1$ scalar superfield Φ and an $\mathcal{N} = 1$ chiral superfield W_γ as in (2.6).
The remaining $\mathcal{N} = 1$ algebra without $\tilde\theta$ is

$$\begin{aligned} \left\{\nabla_a, \bar{\nabla}_{\dot\alpha}\right\} &= i\nabla_{\alpha\dot\alpha} & \left[\nabla_\alpha, \nabla_{\beta\dot\beta}\right] &= C_{\alpha\beta}\bar{W}_{\dot\beta} \\ \{\nabla_\alpha, \nabla_\beta\} &= 0 & \left[\bar{\nabla}_\alpha, \nabla_{\beta\dot\beta}\right] &= C_{\dot\alpha\dot\beta}W_\beta \\ \left\{\bar{\nabla}_{\dot\alpha}, \bar{\nabla}_{\dot\beta}\right\} &= 0 & \left[\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}\right] &= C_{\dot\alpha\dot\beta}\nabla_\beta W_\alpha + C_{\alpha\beta}\nabla_{\dot\beta}W_{\dot\alpha}, \end{aligned} \tag{2.8}$$

which using (2.5) gives us the field strengths . Actually we should distinguish between the field strength multiplet as a superfield and the usual physical field strength which is its θ independent part, but it will be clear from context which one we mean. We note here that from (2.8) and (2.5) one can find the usual relation between the field strength (more precisely its θ-independent part) and the gauge field

$$F_{\underline{a}\underline{b}}\big| = \partial_{\underline{a}}A_{\underline{b}} - \partial_{\underline{b}}A_{\underline{a}} - i[A_{\underline{a}}, A_{\underline{b}}].$$

From (2.8), we see that the field strength can be expressed in terms of the bispinor $f_{\alpha\beta}$ as

$$F_{\alpha\dot\alpha,\beta\dot\beta} = C_{\dot\alpha\dot\beta}f_{\alpha\beta} + C_{\alpha\beta}f_{\dot\alpha\dot\beta}.$$

2.2.2 R-Symmetry

SU(2) R-Symmetry

The $\mathcal{N} = 2$ algebra admits an $SU(2)$ R-symmetry, i.e. the $\mathcal{N} = 2$ superalgebra is invariant under an $SU(2)$ transformation under which the fermionic coordinates θ_α, $\tilde\theta_\alpha$ form a doublet

$$\begin{pmatrix} \theta_\alpha \\ \tilde\theta_\alpha \end{pmatrix} \to U \begin{pmatrix} \theta_\alpha \\ \tilde\theta_\alpha \end{pmatrix}, \qquad U \in SU(2).$$

The central charge is invariant, from which we can find the transformation properties of the physical fields given by (2.6), (2.2), (2.4). It turns out that the scalar ϕ and the bispinor $f_{\alpha\beta}$ transform trivially, the spinors ψ_α, $-\lambda_\alpha$ form a doublet and the fields F, D have more complicated transformation properties. It will be shown that these two fields are only auxiliary fields with no physical significance, therefore we do not have to worry too much about them.

$\mathbf{U(1)}_J$ R-Symmetry

The theory admits also two $U(1)$ symmetries. One of them, denoted as $U(1)_J$ is actually part of the $SU(2)$ R-symmetry. It transforms the fermionic generators and leaves the chiral $\mathcal{N}=2$ field W invariant

$$\begin{aligned}\theta_\alpha &\to \theta'_\alpha = \mathrm{e}^{i\gamma}\theta_\alpha \\ \tilde{\theta}_\alpha &\to \tilde{\theta}'_\alpha = \mathrm{e}^{-i\gamma}\tilde{\theta}_\alpha \\ W(\theta,\tilde{\theta}) &\to W'(\theta',\tilde{\theta}') = W(\theta,\tilde{\theta}).\end{aligned}$$

The spinorial physical fields transform while the scalar and the bispinor remain invariant

$$\begin{aligned}\psi_\alpha &\to \psi'_\alpha = \mathrm{e}^{-i\gamma}\psi_\alpha & \qquad \phi &\to \phi' = \phi \\ \lambda_\alpha &\to \lambda'_\alpha = \mathrm{e}^{i\gamma}\lambda_\alpha & \qquad f_{\alpha\beta} &\to f'_{\alpha\beta} = f_{\alpha\beta}.\end{aligned}$$

Although both the path integral measure for ψ_α and for λ_α are anomalous, these anomalies differ only by a sign and therefore cancel each other. The difference in sign comes from the ψ_α and λ_α have opposite charges. As a result this is an exact quantum symmetry.

$\mathbf{U(1)}_R$ R-Symmetry

The other U(1) symmetry denoted as $U(1)_R$, transforms the fermionic coordinates θ and $\tilde{\theta}$ with the same chargeand transforms the chiral superfield with charge 2,

$$\begin{aligned}\theta_\alpha &\to \theta'_\alpha = \mathrm{e}^{i\gamma}\theta_\alpha \\ \tilde{\theta}_\alpha &\to \tilde{\theta}'_\alpha = \mathrm{e}^{i\gamma}\tilde{\theta}_\alpha \\ W(\theta,\tilde{\theta}) &\to W'(\theta',\tilde{\theta}') = \mathrm{e}^{2i\gamma}W(\theta,\tilde{\theta}) = \mathrm{e}^{2i\gamma}W(\mathrm{e}^{-i\gamma}\theta'_\alpha, \mathrm{e}^{-i\gamma}\tilde{\theta}'_\alpha).\end{aligned}$$

The physical components transform

$$\begin{aligned}\phi &\to \phi' = \mathrm{e}^{2i\gamma}\phi \\ \psi_\alpha &\to \psi'_\alpha = \mathrm{e}^{i\gamma}\psi_\alpha \\ \lambda_\alpha &\to \lambda'_\alpha = \mathrm{e}^{i\gamma}\lambda_\alpha \\ f_{\alpha\beta} &\to f'_{\alpha\beta} = f_{\alpha\beta}.\end{aligned}$$

At the quantum level this symmetry is broken by an anomaly. In this case the transformation of both spinor fields is the same. Therefore the anomalies induced by ψ_α and λ_α are the same and do not cancel but add up to an anomalous transformation of the path integral measure of the spinors

$$\mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\lambda\mathcal{D}\bar{\lambda} \to \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\lambda\mathcal{D}\bar{\lambda}\mathrm{e}^{i8\nu\alpha}$$

with ν being the winding number, which will be defined in the context of monopoles.

2.2.3 Action

Supersymmetric actions are conveniently expressed in superspace. We can define an integral over a Grassmann variable as a linear map that acts on constants and θ as

$$\int \mathrm{d}\theta\ a = 0 \qquad \int \mathrm{d}\theta\ \theta = 1,$$

i.e. it returns the coefficient in front of the θ component. This can of course be generalized to higher dimensional integrals of several Grassmann variables. The integral $\int \mathrm{d}^2\theta^2$ is equivalent to taking the second derivative ∇^2 and then projecting onto $\theta = 0$ up to total derivatives, i.e.

$$\int \mathrm{d}^2\theta^2 g(x,\theta) = \nabla^2 g(x,\theta)\big|_{\theta=0} + \text{total spacetime derivatives.}$$

The supersymmetric action is written in terms of an integral in superspace

$$S = \int \mathrm{d}^4x \quad \int \mathrm{d}^2\theta^2\mathrm{d}^2\bar{\theta}^2\mathrm{d}^2\tilde{\theta}^2\mathrm{d}^2\bar{\tilde{\theta}}^2\mathcal{L}(x,\theta,\bar{\theta},\tilde{\theta},\bar{\tilde{\theta}}).$$

From this we can find the actions for the physical fields. The most general $\mathcal{N} = 2$ action for a chiral superfield with no more than two spacetime derivatives is given by a holomorphic function $\mathcal{F}$ called the prepotential

$$S = \mathcal{I}m \int \mathrm{d}^4x\, \mathrm{d}^2\,\theta\mathrm{d}^2\tilde{\theta}\,\mathcal{F}(W). \tag{2.9}$$

The action is invariant under $U(1)_J$ since θ and $\tilde{\theta}$ transform with opposite charge and the central charge is invariant. It is also invariant under $SU(2)_R$ because the combination $\theta^2\tilde{\theta}^2$ is invariant and the central charge does not transform either. It is however in general not invariant under $U(1)_R$, since it transforms

$$\int \mathrm{d}^2\theta\mathrm{d}^2\tilde{\theta}\mathcal{F}(W(\theta,\tilde{\theta})) \to \int \mathrm{d}^2\theta\mathrm{e}^{-2i\alpha}\mathrm{d}^2\tilde{\theta}\mathrm{e}^{-2i\alpha}\mathcal{F}(\mathrm{e}^{2i\alpha}W(\theta,\tilde{\theta}))$$

The only case when this is invariant is for $\mathcal{F}(W) = W^2$, which corresponds to the classical action.

The action of the component fields

We shall show here in some detail how to obtain the usual action for the component fields.
First we will reduce the $\mathcal{N}=2$ action to $\mathcal{N}=1$ fields. Denoting the derivatives of $\mathcal{F}$ as $\mathcal{F}_A = \frac{\partial \mathcal{F}}{\partial W^A}$, we have

$$\begin{aligned} S &= \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\tilde{\theta} \mathcal{F} = \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, Q^2 \mathcal{F}|_{\tilde{\theta}=0} = \\ &= \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathcal{F}_A Q^2 W^A + \frac{1}{2}\mathcal{F}_{AB} Q^\alpha W^A Q_\alpha W^B \Big|_{\tilde{\theta}=0} = \\ &= \int \mathrm{d}^4x \left\{ \int \mathrm{d}^4\theta \mathcal{F}_A \bar{\Phi}^A + \int \mathrm{d}^2\theta \frac{1}{2} \mathcal{F}_{AB} W^{\alpha A} W^B_\alpha \right\} \end{aligned}$$

where we have used (2.6), (2.7) and (2.8).Reducing this further, we get for the first term

$$\begin{aligned} S_1 &= \int \mathrm{d}^4x \, \nabla^2 \bar{\nabla}^2 \left(\mathcal{F}_A \bar{\Phi}^A \right)|_{\theta=0} = \int \mathrm{d}^4x \, \nabla^2 \left(\mathcal{F}_A \bar{\nabla}^2 \bar{\Phi}^A \right)|_{\theta=0} = \\ &= \int \mathrm{d}^4x \, \frac{1}{2} \mathcal{F}_{ABC} \nabla^\gamma \Phi^C \nabla_\gamma \Phi^B \bar{\nabla}^2 \bar{\Phi}^A \Big|_{\theta=0} + \mathcal{F}_{AB} \nabla^2 \Phi^B \bar{\nabla}^2 \bar{\Phi}^A|_{\theta=0} + \\ &\quad + \mathcal{F}_{AB} \nabla^\gamma \Phi^B \nabla_\gamma \bar{\nabla}^2 \bar{\Phi}^A|_{\theta=0} + \mathcal{F}_A \nabla^2 \bar{\nabla}^2 \bar{\Phi}^A|_{\theta=0} = \\ &= \int \mathrm{d}^4x \, \frac{1}{2} \mathcal{F}_{ABC} \psi^{\gamma C} \psi^B_\gamma F^{\dagger A} + \mathcal{F}_{AB} F^B F^{\dagger A} + \\ &\quad + \mathcal{F}_{AB} \psi^{\gamma B} \left(i \nabla^{\dot{\alpha}}_\gamma \bar{\psi}^A_{\dot{\alpha}} + i[\lambda_\gamma, \bar{\phi}]^A \right) + \mathcal{F}_A \left([D, \bar{\phi}]^A + i\{\bar{\lambda}_{\dot{\beta}}, \bar{\psi}^{\dot{\beta}}\}^A + \Box \bar{\phi}^A \right), \end{aligned} \tag{2.10}$$

where we used the algebra (2.8). The second term is

$$\begin{aligned} S_2 &= \int \mathrm{d}^4x \, \nabla^2 \left(\frac{1}{2} \mathcal{F}_{AB} W^{\alpha A} W^B_\alpha \right) \Big|_{\theta=0} = \\ &= \int \mathrm{d}^4x \, \frac{1}{4} \mathcal{F}_{ABCD} \nabla^\gamma \phi^D \nabla_\gamma \phi^C W^{\alpha A} W^B_\alpha \Big|_{\theta=0} + \\ &\quad + \frac{1}{2} \mathcal{F}_{ABC} \nabla^2 \phi^C W^{\alpha A} W^A_\alpha \Big|_{\theta=0} - \mathcal{F}_{ABC} \nabla^\gamma \phi^C W^{\alpha A} \nabla_\gamma W^B_\alpha|_{\theta=0} - \\ &\quad - \frac{1}{2} \mathcal{F}_{AB} \nabla^\gamma W^{A\alpha} \nabla_\gamma W^B_\alpha \Big|_{\theta=0} + \mathcal{F}_{AB} W^{A\alpha} \nabla^2 W^B \alpha|_{\theta=0} = \\ &= \int \mathrm{d}^4x \, \frac{1}{4} \mathcal{F}_{ABCD} \psi^{\gamma D} \psi^C_\gamma \lambda^{A\alpha} \lambda^B_\alpha + \frac{1}{2} \mathcal{F}_{ABC} F^C \lambda^{\alpha A} \lambda^B_\alpha - \\ &\quad - \mathcal{F}_{ABC} \psi^{\gamma C} \lambda^{\alpha A} (f_{\gamma \alpha^B} + i D^B C_{\gamma\alpha}) + \mathcal{F}_{AB} \lambda^{A\alpha} \nabla^{\dot{\alpha}}_\alpha \bar{\lambda}^B_{\dot{\alpha}} - \\ &\quad - \frac{1}{2} \mathcal{F}_{AB} (f^{\gamma\alpha A} - i D^A C^{\gamma\alpha})(f^B_{\gamma\alpha} - i D^B C_{\gamma\alpha}). \end{aligned} \tag{2.11}$$

We can integrate the term with $\Box = \frac{1}{2} \nabla^{\alpha\dot{\alpha}} \nabla_{\alpha\dot{\alpha}}$ per partes to get the usual kinetic term

$$\int \mathrm{d}^4x \mathcal{F}_A \Box \bar{\phi}^A = - \int \mathrm{d}^4x \, \frac{1}{2} \mathcal{F}_{AB} \nabla^{\alpha\dot{\alpha}} \phi^B \nabla_{\alpha\dot{\alpha}} \bar{\phi}^A. \tag{2.12}$$

We can also find a more familiar form for the kinetic term of the gauge field. From (2.9) we see that

$$\begin{aligned} F^{\alpha\dot{\alpha},\beta\dot{\beta}}F_{\alpha\dot{\alpha},\beta\dot{\beta}} &= 2f^{\alpha\beta}f_{\alpha\beta} + 2f^{\dot{\alpha}\dot{\beta}}f_{\dot{\alpha}\dot{\beta}} = 4\mathcal{R}e\,(f^{\alpha\beta}f_{\alpha\beta}) \\ F^{\underline{ab}}\epsilon_{\underline{abcd}}F^{\underline{cd}} &= (C^{\alpha\beta}f^{\dot{\alpha}\dot{\beta}} + C^{\dot{\alpha}\dot{\beta}}f^{\alpha\beta})(C^{\gamma\delta}f^{\dot{\gamma}\dot{\delta}} + C^{\dot{\gamma}\dot{\delta}}f^{\gamma\delta})i \\ &\quad (C_{\alpha\delta}C_{\beta\gamma}C_{\dot{\alpha}\dot{\beta}}C_{\dot{\gamma}\dot{\delta}} - C_{\alpha\beta}C_{\gamma\delta}C_{\dot{\alpha}\dot{\delta}}C_{\dot{\beta}\dot{\gamma}}) = \\ &\quad = i(2f^{\alpha\beta}2f^{\gamma\delta}C_{\alpha\delta}C_{\beta\gamma} - 2f^{\dot{\alpha}\dot{\beta}}f^{\dot{\gamma}\dot{\delta}}C_{\dot{\alpha}\dot{\delta}}C_{\dot{\beta}\dot{\gamma}}) = \\ &\quad = -8\,\mathcal{I}m\,(f^{\alpha\beta}f_{\alpha\beta}), \end{aligned}$$

here we used $f^{\alpha\beta\dagger} = f^{\dot{\alpha}\dot{\beta}}$. The dual of the field strength tensor is defined as usual $*F_{\underline{ab}} = 1/2\epsilon_{\underline{abcd}}F^{\underline{cd}}$ with $\epsilon_{0123} = 1$. The term $f^{\alpha\beta}f_{\alpha\beta}$ can thus be written in terms of the field strength and its dual

$$f^{\alpha\beta A}f^{B}_{\alpha\beta} = \frac{1}{4}\left(F^{\underline{ab}A}F_{\underline{ab}B} - iF^{\underline{ab}A} * F^{b}_{\underline{ab}}\right),$$

The fields F and D do not appear with derivatives. Using their field equations, which are only algebraic, we can eliminate F and D from the action. Thus these fields are not physical but only auxiliary.

Classical Action

The "classical" action corresponds to the choice of prepotential

$$\mathcal{F} = \frac{1}{4\pi}\tau\sum_{A} W^{A}W_{A}, \tag{2.13}$$

where τ is a complex gauge coupling, given by the usual coupling constant g and the theta angle θ

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}. \tag{2.14}$$

Thus the terms with more than two derivatives drop out, $\mathcal{F}_{AB} = \tau\delta_{AB}$. Inserting everything in (2.10), (2.11) we find

$$\begin{aligned} S = \int \mathrm{d}^4x &- \frac{1}{4g^2}F^{\underline{ab}A}F_{\underline{ab}_A} - \frac{1}{g^2}\nabla^{\underline{a}}\phi^{A}\nabla_{\underline{a}}\phi_{A} + \\ &+ \frac{i}{g^2}\lambda^{\alpha A}\nabla^{\dot{\alpha}}_{\alpha}\bar{\lambda}^{A}_{\dot{\alpha}} + \frac{i}{g^2}\psi^{\alpha A}\nabla^{\dot{\alpha}}_{\alpha}\bar{\psi}^{A}_{\dot{\alpha}} - \\ &- \frac{1}{g^2}f^{ABC}(\bar{\phi}^{A}\psi^{B\alpha}\lambda^{C}_{\alpha} + \phi^{A}\bar{\psi}^{\dot{\alpha}B}\lambda^{C}_{\dot{\alpha}}) - \\ &- F^{A}F^{+A} + D^{A}D^{A} + D^{A}[\bar{\phi}, \phi^{A}] - \\ &- \int \mathrm{d}^4x \frac{\theta}{32\pi^2}F^{A\underline{ab}} * F^{A}_{\underline{ab}}. \end{aligned} \tag{2.15}$$

The terms on the first two lines are the usual kinetic terms of a gauge field, scalar field and spinor field. The terms on the third line are interaction terms for the scalar and

the spinors. The terms on the next line are called F-term and D-term, respectively, and form a potential for the scalar field. Substituting from their equations of motion $F^A = 0$ and $D^A = 1/2[\bar{\phi}, \phi]^A$ we find the positive definite potential

$$V = \frac{1}{2g^2}[\phi, \bar{\phi}]^A[\phi, \bar{\phi}]^A.$$

The last line in (2.15) is in fact a surface term and can be written in terms of the second Chern number c_2 (also called topological charge or instanton number) as

$$\frac{\theta}{32\pi^2}\sum_A \int \mathrm{d}^4x F^{\underline{ab}A}\tilde{F}^A_{\underline{ab}} = \theta c_2 \tag{2.16}$$

where c_2, an integer, is a topological invariant characterizing the gauge bundle (for more on a geometric description of YM theories see [16])

$$c_2 = \frac{1}{8\pi^2}\mathrm{Tr}\int \mathcal{F}\wedge\mathcal{F}.$$

The equations of motion for the physical fields are

$$\begin{aligned}
-\frac{1}{2}\nabla_{\underline{a}}\nabla^{\underline{a}}\phi &= \frac{1}{2}[\phi,[\phi,\bar{\phi}]] + i\{\psi^\gamma, \lambda_\gamma\} \\
i\nabla^{\dot{\gamma}}_{\alpha}\psi^\alpha &= -[\bar{\lambda}^{\dot{\gamma}}, \phi] \\
i\nabla^{\dot{\gamma}}_{\alpha}\lambda^\alpha &= -[\bar{\psi}^{\dot{\gamma}}, \phi] \\
\nabla^{\dot{\alpha}}_{\beta} f^{\alpha\beta} = \frac{1}{2}\nabla_{\beta\dot{\beta}}F^{\alpha\dot{\alpha},\beta\dot{\beta}} &= \frac{1}{2}\left([\phi, \nabla^{\alpha\dot{\alpha}}\bar{\phi}] + [\bar{\phi}, \nabla^{\alpha\dot{\alpha}}\phi]\right) + \{\bar{\psi}^{\dot{\alpha}}, \psi^\alpha\} + \{\bar{\lambda}^{\dot{\alpha}}, \lambda^\alpha\}.
\end{aligned} \tag{2.17}$$

Chapter 3

Classical Monopoles

In this chapter we shall discuss BPS monopoles. For details from the field theoretic view see [17] or [18], for a mathematical viewpoint see [16]. Magnetic monopoles are topologically nontrivial field configurations. Geometrically speaking, we model the Yang-Mills field (the generalization of the electromagnetic field) as a connection on a principal bundle over Minkowski space and by "topologically nontrivial" we mean a nontrivial bundle. Since this material is usually covered without the supersymmetric background, we will change the notation of four-vectors from the dotted-undotted pair $(\alpha\dot{\alpha})$ used in supersymmetry to the usual Greek index μ, and work from now on in the usual framework of fourvectors in QFT.

Dirac monopole

We shall illustrate some ideas on the simplest monopole, the Dirac monopole in electromagnetism. The Dirac monopole [3] is the magnetic equivalent of a point charge, i.e. it has no electric field and a magnetic field

$$\vec{B} = \frac{b}{r^2}, \tag{3.1}$$

with b a real number called the magnetic charge. In terms of the U(1) bundle over the sphere S^2, this cannot be described by only one global formula for A but we must use (at least) two local descriptions. Conventionally these are defined on the northern, resp. southern hemisphere by

$$A_N = b(1 - \cos\theta)\mathrm{d}\theta$$
$$A_S = b(1 + \cos\theta)\mathrm{d}\theta.$$

This solution is called the Wu-Yang monopole [4]. The first Chern number c_1 of this bundle is the integral of the cohomology class of the curvature (field strength) F over the sphere

$$c_1 = \frac{1}{2\pi}\int_{S_2} F = 2b. \tag{3.2}$$

It can be shown, that the Chern number is an integer. From this we find the Dirac quantization condition, which shows that the magnetic charge is quantized

$$b = \frac{n}{2}, \qquad n \in \mathbb{Z}. \tag{3.3}$$

In general if a theory has a compact U(1) gauge group, it has magnetic monopoles with quantized magnetic charges.

3.1 SU(2) Monopoles

We shall now turn to monopoles in Yang-Mills-Higgs theories and we shall start with the simplest gauge group - $SU(2)$. These can be derived from supersymmetric Yang-Mills-Higgs theories by setting the fermionic fields ψ and λ to zero. The results in this section hold also for arbitrary potential, not only the special, supersymmetric case $[\phi, \bar{\phi}]^2$.

3.1.1 SU(2) Yang-Mills-Higgs Action

We will separate the temporal and spatial parts of the fields, the Latin indices i, j, k run over the spatial coordinates $1, 2, 3$). The action with all fermionic terms dropped and written in terms of the electric and magnetic field, $E_i^A = F_{0i}^A$ and $B_i^A = \frac{1}{2}\epsilon_{ijk}F^{jkA}$ is

$$S = \int \mathrm{d}^4x \left[-\frac{1}{2g^2}(B_i^A B_i^A - E_i^A E_i^A) + \frac{\theta}{16\pi^2} E_i^A B_i^A - \right.$$
$$\left. - \frac{1}{g^2}\nabla^\mu \phi^A \nabla_\mu \bar{\phi}^A - \frac{1}{2g^2}([\phi, \bar{\phi}]^A [\phi, \bar{\phi}]^A) \right].$$

We can define the conjugate momenta

$$\Pi_A = \frac{\delta S}{\delta \partial_0 \phi^A} = -\frac{1}{g^2}\nabla^0 \bar{\phi}^A$$
$$\Pi_{iA} = \frac{\delta S}{\delta \partial_0 A_i^A} = \frac{1}{g^2}E_i^A - \frac{\theta}{16\pi^2}B_i^A.$$

Then the Hamiltonian is

$$\mathcal{H} = \Pi_{iA}\partial_0 A_i^A + \Pi_A \partial_0 \phi^A + \bar{\Pi}_A \partial_0 \bar{\phi}^A - \mathcal{L} =$$
$$= \frac{1}{2g^2}(E_i^A E_i^A + B_i^A B_i^A) + \frac{1}{g^2}\nabla_0 \phi^A \nabla_0 \bar{\phi}^A + \frac{1}{g^2}\nabla_i \phi^A \nabla_i \bar{\phi}^A +$$
$$+ \frac{1}{2g^2}([\phi, \bar{\phi}])^2 + A_0^A(-\nabla_i \Pi_i^A + \frac{1}{g^2} f^{ABC}(\nabla_0 \bar{\phi}^B \phi^C + \nabla_0 \phi^B \bar{\phi}^C))$$

where we rewrote everything in terms of covariant objects and then integrated per partes the term involving $\nabla_i A_0^A$. Since the momentum conjugate to A_0 is zero, we must impose that this holds for all times, i.e. we must impose that the Poisson bracket of A_0 and the Hamiltonian vanishes. This gives rise to the Gauss constraint

$$\nabla_i \Pi_i^A = \frac{1}{g^2} f^{ABC}(\nabla_0 \bar{\phi}^B \phi^C + \nabla_0 \phi^B \bar{\phi}^C)).$$

We will work in the gauge $\nabla_0 \phi^A = 0$, then the Gauss constraint becomes (taking into account the Bianchi identity $\nabla_i B_i^A = 0$)

$$\nabla_i \Pi_i^A = \nabla_i E_i^A.$$

The vacuum is the state with zero energy; it must satisfy the equations

$$E_i^A = B_i^A = 0 \qquad \nabla_i \phi^A = 0 \qquad [\phi, \bar{\phi}] = 0. \tag{3.4}$$

The vacuum expectation value of ϕ can be written as $\langle \phi^A \rangle = a\mathrm{e}^A$ with e^A being a unit vector in the algebra. The condition on ϕ determines a variety of gauge equivalent configurations called the Higgs vacuum. In our example with SU(2) the Higgs vacuum is given by $\phi^2 = a^2$, so the values of the Higgs vacuum form a sphere in the space all configurations.This breaks the SU(2) down to U(1), which is generated by $T = T^A \mathrm{e}^A$ and corresponds to rotations of ϕ with rotation axes along $\langle \phi \rangle$. The gauge field separates into two components: the longitudinal component $A_\mu^{\|}$ which is parallel (in the Lie algebra) to $\langle \phi \rangle$ remains massless and is the U(1) gauge field, and the two perpendicular components which form a selfinteracting complex field $A_\mu^{\pm} = A_\mu^{\perp} \pm i\tilde{A_\mu^{\perp}}$ with charge ± 1 and mass ag.
Monopoles are finite energy configurations so they must approach the vacuum at infinity. The scalar field induces a map from the sphere at infinity S^2_∞ to the space of Higgs vacua, which is also a sphere S^2_a but with radius a. All maps from S^2 to a target manifold M can be characterized by their homotopy class. For $M = S^2_a$ this is $\mathbb{Z}$, so the scalar field can be characterized by an integer. This integer is called winding number, and represents how many times S^2_∞ wraps around S^2_a [1].
The winding number is directly related to the magnetic charge of a given solution.It follows from (3.4), that the perpendicular component of the gauge field must be

$$A_\mu^{\perp} = -\frac{i}{a^2}[\partial_\mu \phi, \phi].$$

The longitudinal component of the field strength is

$$F_{\mu\nu}^{\|} = \partial_\mu A_\nu^{\|} - \partial_\nu A_\mu^{\|} - \frac{i}{a^3}(\phi^A [\partial_\nu \phi, \partial_\mu \phi]^A)$$

Now compute the magnetic charge of the U(1) field, which is the surface integral of the magnetic field $B_i = \frac{1}{2}\epsilon^{ijk} F_{jk}^{\|}$ divided by 4π (the 4π are added so that the definition agrees with (3.1)),

$$\begin{aligned} b &= \frac{1}{4\pi} \int_{S^2_\infty} B_i^{\|} \mathrm{d}S_i = \\ &= \frac{1}{4\pi} \int_{S^2_\infty} \left(\frac{1}{2}\epsilon^{ijk}(\partial_j A_\kappa^{\|} - \partial_k A_j^{\|}) - \frac{i}{2a^3}\epsilon^{ijk}(\phi^A [\partial_k \phi, \partial_j \phi]^A \right) \mathrm{d}S_i = \\ &= \frac{1}{4\pi} \int_{S^2_\infty} \left((\mathrm{rot} A)_i - \frac{1}{2a^3}\epsilon^{ijk}\epsilon^{ABC}\phi^A \partial_j \phi^B \partial_k \phi^C \right) \mathrm{d}S_i. \end{aligned}$$

The part involving the rotation vanishes by Stokes' theorem since S^2_∞ has no boundary. For the second term, note that the volume form on the Higgs vacuum is

$$\tilde{\omega} = \frac{1}{2a}\epsilon^{ABC}\phi^A \mathrm{d}\phi^B \wedge \mathrm{d}\phi^C$$

[1] Definition: Let $\phi : M \to V$ be a smooth map from one closed oriented manifold (ie. compact, without boundary) to another of the same dimension n. Let ω be a normalized n-form on V, ie. $\int_V \omega = 1$. The winding number of ϕ is the integer $\nu = \int_M \phi^* \omega$.

and the normalized n-form $\omega = \tilde{\omega}/(4\pi a^2)$. The pullback of ω is then

$$\phi^*\omega = \frac{1}{8\pi a^3}\epsilon^{ABC}\phi^A \frac{\partial\phi^B}{\partial x^j}\frac{\partial\phi^C}{\partial x^k}\mathrm{d}x^j \wedge \mathrm{d}x^k$$

with $\mathrm{d}x^j \wedge \mathrm{d}x^k = \epsilon^{ijk}\mathrm{d}S_i$. This gives the second term, hence

$$b = -\nu,$$

where ν is the winding number.

3.1.2 BPS Monopoles

The Hamiltonian for the static configuration can be split into two parts; one is positive definite and the other is a surface term

$$\begin{aligned} H &= \frac{1}{2g^2}\int \mathrm{d}^3x \left(B_i^A B_i^A + E_i^A E_i^A + 2\nabla_i\phi^A\nabla_i^A\bar{\phi}^A + V(\phi)\right) = \\ &= \frac{1}{2g^2}\int \mathrm{d}x^3 \left|B_i^A + iE_i^A + \sqrt{2}\mathrm{e}^{i\alpha}\nabla_i\phi^A\right|^2 + V(\phi) - \\ &- \frac{\sqrt{2}}{g^2}\int_{S^2_\infty} \mathrm{d}S_i \left(B_i^A \mathcal{R}e\left(\mathrm{e}^{i\alpha}\phi^A\right) + E_i^A \mathcal{I}m\,\mathrm{e}^{i\alpha}\phi^A\right), \end{aligned}$$

$V(\phi)$ is the potential for the scalar field and $\mathrm{e}^{i\alpha}$ is a phase, that has been included to get the most general form possible. We have used $\nabla_i B_i^A = 0$ and $\nabla_i E_i^A = 0$ and written the integral as a surface term. The first term is positive definite (α is an arbitrary parameter), it is zero if $V(\phi) = 0$ and

$$B_i^A + iE_i^A + \sqrt{2}\mathrm{e}^{i\alpha}\nabla_i\phi^A = 0. \tag{3.5}$$

This is called the Bogomol'nyi equation and solutions to it are called BPS (Bogomol'nyi-Prasad-Sommerfeld) monopoles. It can be shown that any configuration which solves the BPS equation also solves the equations of motion. The second term is a surface term and constitutes a lower bound to the energy. In terms of the magnetic charge b and electric charge q (of the residual U(1))

$$E \geq \frac{\sqrt{2}}{g^2}(b\cos\alpha - q\sin\alpha).$$

We can define the magnetic and electric quantum numbers n_m and n_e by

$$n_m a_D = -\frac{1}{4\pi}\int_{S^2_\infty} \mathrm{d}S_i \tau B_i^{\|} a \qquad n_e a = -\int_{S^2_\infty} \mathrm{d}S_i \Pi_i^{\|} a,$$

with $a_D = \tau a$. Then the surface term can be written as

$$H_2 = \sqrt{2}\,\mathcal{I}m\,\mathrm{e}^{i\alpha}(n_m a_D + n_e a).$$

In fact, the central charge of this theory can be shown to be [19]

$$Z = n_m a_D + n_e a, \tag{3.6}$$

so in order to reproduce the algebraic energy bound (2.1) we must set α to $\alpha = \pi/2 - \arg Z$.
Comparing the two expressions for H_2 we see also that

$$b = -4\pi n_m \qquad\qquad q = -n_e + n_m \frac{\theta}{2\pi},$$

i.e. if the theta angle is nonzero, a magnetic charge has also electric charge. This is also known as the Witten effect.

3.1.3 t'Hooft-Polyakov Monopoles

The 't Hooft-Polyakov monopole [8], [20] is an exact, static solution to the Bogomol'nyi equation (3.5). It can be found by making a particular ansatz, for which we shall give some motivation.
The theory has two $SO(3)$ symmetries. One comes from the spatial part of the Lorentz group and one from the $SU(2)$ gauge group which is a covering of $SO(3)$ $\pi : SU(2) \to SO(3)$. The solution can be neither rotationally invariant since then it would have zero topological charge nor gauge invariant. However, we can combine the rotational symmetry $SO(3)_R$ and the gauge symmetry $SU(2)_G$, and require that the solution be invariant under the diagonal subgroup of $SO(3)_R \times SO(3)_G$. We can also add a $\mathbb{Z}_2$ symmetry which consists of parity plus sign change of the Higgs field and the temporal component of the connection A_0.
The action of $SO(3)_R$ on the position vector $\vec{x}$ is ordinary matrix multiplication $g\vec{x}$, the action of $SO(3)_G$ on the scalar and vector field is the adjoint action of SU(2) $\mathrm{Ad}(g)(\phi)$. The 't Hooft-Polyakov ansatz thus requires the fields to be invariant under the following transformations [2]

$$\begin{aligned}
&\mathrm{diag}(SO(3)_R \times \pi(SU(2)_G)) \ni g : \\
&\qquad A_j(\vec{x}) \to A_j(g\vec{x}) = \mathrm{Ad}(g)(A_i(\vec{x}))\left(g^{-1}\right)^i_{\ j} \\
&\qquad A_0(\vec{x}) \to A_0(g\vec{x}) = \mathrm{Ad}(g)(A_0(\vec{x})) \\
&\qquad \phi(\vec{x}) \to \phi(g\vec{x}) = \mathrm{Ad}(g)(\phi(\vec{x})) \\
&\mathbb{Z}_2 : \\
&\qquad A_j(\vec{x}) \to A_j(-\vec{x}) = -A_j(\vec{x}) \\
&\qquad A_0(\vec{x}) \to A_0(-\vec{x}) = -A_0(\vec{x}) \\
&\qquad \phi(\vec{x}) \to \phi(-\vec{x}) = -\phi(\vec{x}).
\end{aligned}$$

Expanding the potential A in a basis of the algebra we have for the coefficients

$$\begin{aligned}
A^A_j(\vec{x})T_A &= gA^A_i(g^{-1}\vec{x})T_A g^{-1}\left(g^{-1}\right)^i_{\ j} = \\
&= A^A_i(g^{-1}\vec{x})\left(g^{-1}\right)_A^{\ C}\left(g^{-1}\right)^i_{\ j} T_C
\end{aligned}$$

[2] Technically, the map $\pi : SU(2) \to SO(3)$ is 2:1 and we should check whether the transformation is well defined. We can choose the map π so that the pair g, $-g$ always gets mapped on the same element $[g]$. In the transformation itself there are always two elements from SU(2) g and g^{-1}, the other g, if present, is actually the image $[g]$. Therefore, the minus signs cancel.

where we used the fact that the relation $gT_Ag^{-1} = \left(g^{-1}\right)_A{}^B T_B$ holds for the su(2) generators T_A . We see that the coefficients A_i^A transform according to the $3 \otimes 3$ representation:

$$A_i^A(\vec{x}) = A_j^B(g^{-1}\vec{x}) \left(g^{-1}\right)^j{}_i \left(g^{-1}\right)_B{}^A .$$

Thus A_i^A must be from $6 \oplus 3$ and can be written in terms of two unknown functions ($r^2 = \sum (x^i)^2$)

$$A_i^A = \alpha(r)_i^A + \beta(r)\epsilon^A{}_{ik}x^k.$$

The function α_i^A is even under parity and the β term is odd. Thus imposing the $\mathbb{Z}_2$ we must set α to zero.
For the Higgs field written in terms of the basis $\phi = \phi^A T_A$, we have the condition

$$\phi^A(\vec{x})T_A = g\phi^A(g^{-1}\vec{x})T_A g^{-1} = \phi^A(g^{-1}\vec{x}) \left(g^{-1}\right)_A T_B.$$

Thus the coefficients ϕ_A transform in the 3 representation

$$\phi^A(\vec{x}) = \left(g^{-1}\right)_B{}^A \phi^B(g^{-1}\vec{x})$$

and the most general form of this is

$$\phi^A(\vec{x}) = \gamma(r)x^A.$$

This has also the correct transformation property with respect to $\mathbb{Z}_2$.
Starting with this Ansatz

$$\phi^A = \mathrm{e}^A\phi(r) \qquad A_i^A = \epsilon^A{}_{ij}\mathrm{e}^j \left(\frac{1 - L(r)}{r}\right) \qquad A_0^A = \mathrm{e}^A b(r) \tag{3.7}$$

where $r = \sqrt{x^i x^i}$ is the usual distance from the origin and $\mathrm{e}^A = x^A/r$ is a unit radial vector, we get the electric and magnetic fields

$$B_i^A = \mathrm{e}_i\mathrm{e}^A \frac{L^2 - 1}{r^2} + \mathcal{P}_i^A \frac{L_{,r}}{r} \tag{3.8}$$

$$E_i^A = -\mathrm{e}_i\mathrm{e}^A b_{,r} - \mathcal{P}_i^A \frac{bL}{r} \tag{3.9}$$

with the projector

$$\mathcal{P}_i^A = \delta_i^A - \mathrm{e}_i\mathrm{e}^A. \tag{3.10}$$

The components proportional to $\mathrm{e}^A\mathrm{e}_i$, resp., to $\mathcal{P}_i^A$ are called Abelian, resp., non-Abelian, since $\mathrm{e}_i\mathrm{e}^A$ projects on the Abelian $U(1)$.
The ansatz is perhaps more intuitive in spherical coordinates: at each point in space we choose the basis vectors of the Lie algebra T_R, T_Θ, T_Φ to be parallel to the ($\mathbb{R}^3$) basis vectors e_r, e_θ, e_φ, resp.

$$\begin{aligned} T_R &= \cos\varphi \sin\vartheta T_1 + \sin\varphi \sin\vartheta T_2 + \cos\vartheta T_3 \\ T_\Theta &= \cos\varphi \cos\vartheta T_1 + \sin\varphi \cos\vartheta T_2 - \sin\vartheta T_3 \\ T_\Phi &= -\sin\varphi T_1 + \cos\varphi T_2. \end{aligned}$$

This implies that the Lie algebra basis vectors depend on the spherical coordinates in space and we thus have a spin connection.
In this basis, we see that the nonzero components of the scalar and gauge field are

$$\phi^R = \phi \qquad A^\Theta_\varphi = -A^\Phi_\vartheta = \frac{1-L}{r} \qquad A^R_0 = b$$

and

$$B^R_r = \frac{L^2-1}{r} \qquad E^R_r = -b_{,r} \qquad \text{(Abelian components)}$$
$$B^\Theta_\vartheta = B^\Phi_\varphi = \frac{L_{,r}}{r} \qquad E^\Theta_\vartheta = E^\Phi_\varphi = -\frac{bL}{r} \qquad \text{(non-Abelian components).}$$

Calculating also the covariant derivative of ϕ, we find

$$\nabla\phi = \phi_{,r}T^R\mathrm{d}r + \phi LT^\Theta\mathrm{d}\vartheta + \phi L\sin\theta T^\Phi\mathrm{d}\varphi.$$

Inserting in the Bogomol'nyi equation (3.5) we get two independent equations, one for the Abelian component and one for the non-Abelian component

$$\sqrt{2}\mathrm{e}^{i\alpha}\phi_r = \frac{1-L^2}{r^2} + ib_r \qquad \sqrt{2}\mathrm{e}^{i\alpha}\phi = -\frac{\mathrm{d}}{\mathrm{d}r}\ln L + ib. \tag{3.11}$$

Combining these, we get a second order differential equation for L

$$\frac{L_{rr}}{L} - \frac{(L_r)^2}{L^2} = \frac{L^2-1}{r^2} \tag{3.12}$$

which is solved by

$$L = \frac{\kappa r}{\sinh[\kappa(r+\delta)]} \tag{3.13}$$

with integration constants δ and κ. The constant κ is given by the long range behavior of the Higgs field. Inserting

$$\frac{\mathrm{d}\ln L}{\mathrm{d}r} = \frac{1}{r} - \frac{\kappa}{\tanh[\kappa(r+\delta)]}$$

in the Bogomol'nyi equation (3.5) and taking the $r \to \infty$ limit, we find that

$$\kappa = \mathcal{R}e\,(\sqrt{2}\mathrm{e}^{i\alpha}a).$$

If we require the potentials A^A_i to be finite everywhere we get the condition $L \to 1$ for $r \to 0$. Then the parameter δ must be chosen to be zero. The winding number of this solution is -1 and the magnetic charge therefore 4π.

3.2 General Gauge Group Monopoles

The derivation of the Bogomol'nyi equation which was presented in subsections 3.1.1 and 3.1.2 does not depend on the fact that we chose SU(2) as the gauge group. The details of the Higgs vacuum and the classification by using winding numbers is, of

course, only valid for SU(2). Apart from that, all the other algebraic steps can be done for general groups and thus equation (3.5) holds for any gauge group. We shall show how the t'Hooft Polyakov monopole can be used to construct monopoles of unit charge for general groups, [21].
The calculations in this section are simpler in the rescaled basis defined in appendix C.

The choice of simple roots depends on the asymptotic value of the scalar field Φ_0. We can choose this to lie in the Cartan subalgebra and define a vector in the algebra $\vec{h}$ (the equivalent of the value a for SU(2)) by

$$\Phi = \vec{h}\vec{H}. \tag{3.14}$$

Now we require that the simple roots all have nonnegative inner products with $\vec{h}$. If there are no roots orthogonal to $\vec{h}$, there is a unique choice for the simple roots and the symmetry breaking is maximal to $U(1)^r$. If there are k orthogonal roots, these form a sublattice which is the rootlattice of a semisimple subalgebra K. The symmetry breaking is then $U(1)^{r-k} \times K$. The choice of roots is now ambiguous, with the various possibilities related by elements of the Weyl group of K. We can further require the asymptotic magnetic field to lie in the Cartan algebra and be of the form

$$B_i = \frac{\hat{r}^i}{r^2}\vec{g}\vec{H}. \tag{3.15}$$

It has been shown [22] that $\vec{g}$ must fulfill the quantization condition

$$\vec{g} = \sum_a n_a \vec{\beta}_a^* + \sum_j q_j \vec{\gamma}_j^* \tag{3.16}$$

with $\vec{\beta}^*$ the dual root of $\vec{\beta}$, $\vec{\beta}^* = \vec{\beta}/\vec{\beta}^2$ and n_a, q_j integers. The n_a are uniquely determined, gauge invariant topological invariants, the magnetic quantum numbers. For maximal symmetry breaking there is a fundamental monopole corresponding to each simple root and for each composite root there is a multimonopole solution consisting of those monopoles, that correspond to roots forming the composite root. The monopole associated to the root $\vec{\beta}$ can be obtained as follows. First we can define an $su(2)_\beta$ subalgebra associated to the root

$$\begin{aligned} t^1 &= \frac{1}{\sqrt{2\vec{\beta}^2}}(E_\alpha + E_{-\alpha}) \\ t^2 &= -\frac{i}{\sqrt{2\vec{\beta}^2}}(E_\alpha - E_{-\alpha}) \\ t^3 &= \vec{\beta}^*\vec{H}. \end{aligned} \tag{3.17}$$

The Cartan subalgebra separates into a component lying in this $su(2)$ generated by t^3 and an complementary component. This component can be chosen in terms of the

expectation value as $t^{\perp} = \left(\vec{h} - \vec{h}\cdot\vec{\beta^*}\vec{\beta}\right)\cdot\vec{H}$. The t'Hooft-Polyakov monopole (3.7), (3.13) can be embedded in $\mathrm{SU}(2)_\beta$

$$\begin{aligned} A_\mu &= \sum_{s=1}^{3} A_\mu^s t^s \\ \Phi &= \sum_{s=1}^{3} \phi\hat{r}^s t^s + (\vec{h} - \vec{h}\cdot\vec{\beta}\vec{\beta^*})\cdot\vec{H}, \end{aligned} \tag{3.18}$$

with $\kappa = \mathcal{R}e\left(\mathrm{e}^{i\alpha}\vec{h}\vec{\beta}\right)$. This is a solution to the Bogomol'nyi equation and carries the charge n_a, which is the coefficient of the dual root in terms of the dual simple roots $\beta^* = \sum_a n_a\beta_a^*$.
The electric and magnetic fields and the covariant derivative of the scalar field lie in $su(2)_\beta$ and have the same form as in SU(2) with e^A exchanged for t^s:

$$B_i^s = \frac{L_{,r}}{r}(\delta_{is} - \hat{r}^i\hat{r}^s) + \frac{L^2-1}{r^2}\hat{r}^i\hat{r}^s \tag{3.19}$$

$$E_i^s = -b_{,r}\hat{r}^i\hat{r}^s + \frac{bL}{r}(\delta_{is} - \hat{r}^i\hat{r}^s) \tag{3.20}$$

$$\nabla_i\Phi^s = \frac{\phi L}{r}(\delta_{is} - \hat{r}_i\hat{r}_s) + \phi_{,r}\hat{r}_i\hat{r}_s, \tag{3.21}$$

with L, b and ϕ the same as for SU(2).

Chapter 4

Quantum Monopoles

Two papers by Seiberg and Witten [10, 11] started a large amount of work done in the field of quantum monopoles. In these papers, they found the exact form of the effective action for $\mathcal{N} = 2$ supersymmetric Yang-Mills with gauge group SU(2) both with and without matter. Their results were later generalized to other gauge groups by [23–27]. There exist some very good reviews on this topic, see e.g. [28], [29].
The work was later combined with BPS monopoles, [30].

4.1 Seiberg-Witten Theory

As has been mentioned before, $\mathcal{N} = 2$ theories of gauge fields have a very particular form: the action is given by one holomorphic function. Furthermore, as a consequence of supersymmetry, corrections to this superpotential occur only at one loop level. Thus we have more control over the quantum corrections than in $\mathcal{N} = 1$ theories, where not all quantum corrections are restricted to one loop level, but we have a non-trivial beta function, unlike $\mathcal{N} = 4$ where the beta function simply vanishes. As for the choice of gauge group, it is natural to start with the simplest non-Abelian group SU(2).
Quantum corrections are encoded in the effective action. In this case we look for a Wilsonian effective action which could in principle be obtained by integrating out fluctuations above some scale.
This effective action will depend on the symmetry breaking. As mentioned below (3.4), we can choose the vacuum expectation value in the form $\langle\phi\rangle = a\mathrm{e}^A T^A$ with e^A a unit vector. This breaks SU(2) to U(1). Excitations associated with this U(1), i.e. parallel to e^A in the Lie algebra, remain massless, the others acquire a mass ag. A gauge transformation from the Weyl group changes the sign $a \to -a$, so instead we will use the gauge invariant object $u = \mathrm{Tr}\ \phi^2$ which is classically $u = \frac{1}{2}a^2$. At $u = 0$ we expect classically a singularity since the SU(2) symmetry is restored and the W boson becomes massless. In the quantum picture we expect a singularity as usual when fields that are integrated out in the Wilsonian action become massless.It is known from supersymmetry that the effective action receives both perturbative

and non-perturbative corrections and for a U(1) field must be of the form [31]

$$\mathcal{F} = \frac{1}{2}\tau_0 W^2 + \frac{i}{2\pi} W^2 \log\left[\frac{W^2}{\Lambda^2}\right] + \frac{1}{2\pi i} W^2 \sum_{l=1}^{\infty} c_l \left(\frac{\Lambda}{W}\right)^{4l} \tag{4.1}$$

where Λ is a dynamically generated scale, τ_0 is the bare coupling, the middle term is the one-loop correction and the last term is the instanton term. From expansions of the action (2.10),(2.11) we know that the second derivative of $\mathcal{F}$ can be interpreted as the effective coupling τ. Obviously, due to the logarithm the effective coupling will not be single valued. This can also be regarded also from a different point of view. From the kinetic term for the scalar field in (2.12), we see that this is a sigma model and thus $\mathcal{I}m\,\tau$ is also the metric on the moduli space. So this requires $\mathcal{I}m\,\tau$ to be positive everywhere. Now, $\mathcal{I}m\,\tau$ is a harmonic function, i.e. $\partial\bar{\partial}\mathcal{I}m\,\tau = 0$ because $\tau(u)$ is a holomorphic function of the modulus u. A theorem from complex analysis states that if a function that is harmonic on a connected open set, has a minimum, it is constant. Thus we see that it can be defined only locally.
Seiberg and Witten proposed that the quantum moduli space should have two singularities at the points $u = \pm\Lambda^2$ and the singularity at infinity. The motivation for this is the following. There is a $\mathbb{Z}_2$ symmetry acting on the moduli space as $u \to -u$. This comes from the $U(1)_R$ defined in section 2.2.2, which is broken by the one-loop term and the instanton terms to $\mathbb{Z}_8$, as can be seen from the following. As shown in section 2.2.2 the prepotential must transform as $\mathcal{F} \to \mathrm{e}^{4i\alpha}\mathcal{F}$ for the action to be invariant. The instanton term transforms

$$W^2 \sum_{l=1}^{\infty} c_l \left(\frac{\Lambda}{W}\right)^{4l} \to \mathrm{e}^{4i\alpha} W^2 \sum_{l=1}^{\infty} c_l \left(\frac{\Lambda}{W}\right)^{4l} \mathrm{e}^{-8li\alpha}$$

so the action is invariant for $\exp(-8il\alpha) = 1$ for any l. But this means that α must be $\frac{2\pi}{8}k$, which is $\mathbb{Z}_8$. The one-loop term transforms

$$\frac{i}{2\pi} W^2 \log\frac{W^2}{\Lambda^2} \to \frac{i}{2\pi}\mathrm{e}^{4i\alpha} W^2 \log\frac{W^2}{\Lambda} - \frac{2\alpha}{\pi} W^2 \mathrm{e}^{4i\alpha},$$

but the second term changes only $\mathcal{R}e\,\tau$ (effectively the θ angle transforms as $\theta \to \theta - 8\alpha$) which is the coefficient for the instanton number (2.14), (2.16). So the action gets changed by $8\alpha c_2$ where c_2 is the instanton number. In a path integral we have $\exp(iS)$, so a shift by a multiple of 2π doesn't change the physics. But this is exactly the requirement that $\alpha = \frac{2\pi}{8}k$. This $\mathbb{Z}_8$ acts on $u = \mathrm{Tr}\,\phi^2$ as $u \to u\,\mathrm{e}^{i4\alpha} = u\,\mathrm{e}^{i4\frac{2\pi}{8}k}$, which is (for odd k) $u \to -u$. This symmetry must also be present in the structure of the moduli space. The only possibility to have only one singularity(plus the one at infinity) would be at $u = 0$. But then the relations for a, a_D and τ would be valid globally and we saw from the analysis of τ that this is not what we want. The second simplest choice is to have two singularities and one singularity at infinity, which turns out to be the right one. In fact, it can be shown that there is no solution for any other arrangement of singularities [32]. Seiberg and Witten postulated that the physical interpretation of these singularities is that at these points certain excitations (dyons) become massless.

Their idea was to study the singularities by means of monodromies. Monodromies describe how a function changes when we loop around a singularity (or branch point). The rigorous definition [1] may seem a little complicated but a well-known example of this is the behavior of the log-function in the complex plane when we loop around the origin $\log z \to \log z + 2\pi i$. As such, monodromies arise frequently in complex analysis and analytic continuation.
Seiberg and Witten defined a second scalar field as the derivative of the prepotential $a_D = \partial\mathcal{F}/\partial a$, which gives classically $a_D = \tau a$. In the semi-classical region, close to infinity, we see that a and a_D are given by

$$\begin{aligned} a_D &= \tau_0 a + \frac{i}{2\pi} 2a \log\frac{a^2}{\Lambda} + \frac{2i}{\pi} a + \text{instanton terms} \\ a &= \sqrt{2u} \end{aligned} \tag{4.2}$$

Performing a loop in the u plane $u \to \mathrm{e}^{i2\pi} u$ we find that

$$\begin{aligned} a &\to -a \\ a_D &\to -a_D + 2a. \end{aligned}$$

This is a monodromy transformation

$$\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \to M \begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix}$$

with monodromy matrix[2] M_∞

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}. \tag{4.3}$$

At the point $u = \Lambda^2$ a monopole becomes massless. Near this point solitonic degrees of freedom become lighter than the fundamental fields which we use in the effective action. We must switch to an alternative description in which the fundamental objects are the solitonic states and their superpartners. These solitonic states are called dual to the fundamental ones. In this sense a_D is the dual field to a.

$$\begin{aligned} a_D &= c(u - \Lambda^2) \\ a &= a_0 + \frac{i}{\pi} c(u - \Lambda^2) \log(u - \Lambda^2). \end{aligned} \tag{4.4}$$

[1] Consider the covering $p : \tilde{X} \to X$ of a connected topological space X with base point x and denote the fiber of the base point as $F = p^{-1}(x)$. For each loop γ based at x, there is a unique lift $\tilde{\gamma}$ such that $p \circ \tilde{\gamma} = \gamma$. The map $F \times \pi_1(X, x) \ni (\tilde{x}, \gamma) \to \tilde{\gamma}(1) \in F$ forms a right group action of the fundamental group $\pi_1(X, x)$ on the fiber F. This action is called monodromy action, the corresponding map $\pi_1(X, x) \to \mathrm{Sym}F$, where $\mathrm{Sym}F$ is the symmetric group of F, is called monodromy, and the image of this map is called monodromy group.

[2] In ordinary differential equations with periodic solutions a monodromy matrix is the fundamental solution matrix evaluated at a time equal to the period and is closely related to questions of stability of solutions.

The monodromy for this when we loop around $u = \Lambda^2$ is

$$\begin{aligned} a_D &\to a_D \\ a &\to a - 2a_D \end{aligned}$$

with monodromy matrix M_{Λ^2}

$$M_{\Lambda^2} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}. \tag{4.5}$$

There is a global consistency condition on monodromies since paths in the complex plane can be distorted. This condition ties together the three monodromies

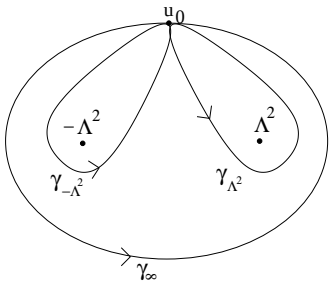

Figure 4.1: *Paths associated with different monodromies*

$$M_{\Lambda^2} \cdot M_{-\Lambda^2} = M_\infty.$$

The two paths γ_1, γ_2 in fig. 4.1 can be joined to form the path $\gamma_\infty \sim \gamma_{\Lambda^2} \circ \gamma_{-\Lambda^2}$. The order of the matrices comes from the fact that the monodromy group is a representation of the right action. The choice of the base point plays an important role in the interpretation of the massless particles' charges as we will see. For a base point in the upper half plane the remaining monodromy matrix is

$$M_{-\Lambda^2} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}.$$

The mass of a particle is given by the central charge $m = \sqrt{2}|Z|$ and the formula (3.6) which can also be written in the form

$$Z = (n_m n_e) \begin{pmatrix} a_D \\ a \end{pmatrix}.$$

So looping around a singularity is equivalent to shifting the electric and magnetic quantum numbers $(n_m\ n_e) \to (n_m\ n_e)M$. A particle that becomes massless at a certain singularity must have quantum numbers that are invariant under this shift. From this we find that the particles which become massless at $\pm\Lambda^2$ are the monopole (1,0) and the dyon (1,-1). At infinity the W boson (0,1) becomes massless. If we change u_0 to $-u_0$ (i.e. if we act with $\mathbb{Z}_2$ we find that the order of the matrices in the multiplication changes. Consequently the monodromy matrix also changes and the

interpretation of the massless particle associated to it. For a base point in the lower half plane we find that a (1,1) dyon becomes massless.
On the other hand this requires the monodromy matrix to be of a particular form, namely

$$M_{(n_m,n_e)} = \begin{pmatrix} 1+2n_m n_e & 2n_e^2 \\ -2n_m^2 & 1-2n_m n_e \end{pmatrix}. \tag{4.6}$$

If there were to be more than three singularities we would have a condition for them analogous to (4.6). But since the monodromy matrices are required to have a very special form, it is a problem of number theory, whether there exist solutions with integer quantum numbers. It has been checked that for more than two strong-coupling singularities there are no such solutions [23, 24].

The problem is now a mathematical one. Namely, to find multi-valued functions $a(u)$ and $a_D(u)$ with the given monodromies and additionally with a coupling $\tau = \partial a_D/\partial a$ such that $\mathcal{I}m\,\tau > 0$. This is a classical mathematical problem, the "Riemann -Hilbert" problem, which is known to have a unique solution (up to multiplication by an entire function). The Riemann-Hilbert problem addresses the question whether it is possible to find a system of differential equations in the complex plane (up to a few points or regular singularities) such that the solutions display the required monodromy group. There are two possible approaches:

1. differential equations, which can be solved and explicit formulas for a and a_D can be found

2. spectral surfaces and period integrals, which offer a more geometric point of view.

4.2 Elliptic Curves

The basic idea of solving the monodromy problem using elliptic curves is to reformulate it in terms of a toroidal Riemann surface with a moduli space equivalent to the quantum moduli space of the original theory. The advantage is that we can interpret the gauge coupling τ as the period matrix of the torus and as such it is guaranteed that $\mathcal{I}m\,\tau > 0$. The torus can be represented as a two-sheeted cover of a branched x-plane. The algebraic expression for such a surface would be $y^2 = f(x,u)$, i.e. the complex moduli u parametrizes the surface in a certain way. There are two independent cycles on a torus see fig. 4.2. The period matrix can be written as the ratio of two integrals

$$\tau = \frac{\varpi_D}{\varpi}$$

where ϖ_D, ϖ are so-called period integrals

$$\varpi_D = \oint_\beta \omega \qquad\qquad \varpi = \oint_\alpha \omega$$

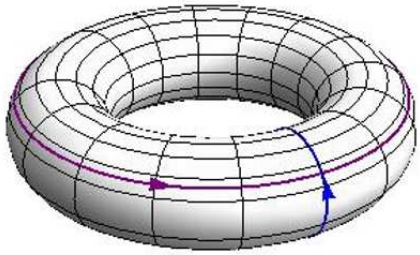

Figure 4.2: *Cycles on a torus*

and ω is a holomorphic differential

$$\omega = \frac{1}{\sqrt{2}\pi}\frac{\mathrm{d}x}{y}. \tag{4.7}$$

Since in the gauge theory $\tau = \partial_a a_D$ we see that the period integrals are related to the scalar and dual field by

$$\varpi_D = \frac{\partial a_D}{\partial u} \qquad\qquad \varpi = \frac{\partial a}{\partial u}. \tag{4.8}$$

Then these fields can be written as integrals of some differential, which is given only up to exact forms

$$a_D = \oint_\beta \lambda_{SW} \qquad\qquad a = \oint_\alpha \lambda_{SW}. \tag{4.9}$$

The singularities of the moduli space are points at which the torus degenerates, i.e. the singularities which form the branches coincide. Cycles, which shrink to zero in such a case, are called vanishing cycles and correspond to massless particles. Since the cycles α, β form a basis, all cycles can be decomposed as $\nu = g\beta + q\alpha$. If this cycle vanishes we find that the central charge (and thus also the mass) vanishes

$$Z = ga_D + qa = g\oint_\beta \lambda_{SW} + q\oint_\alpha \lambda_{SW} = \oint_\nu \lambda_{SW} = 0. \tag{4.10}$$

We see also that the coefficients of the decomposition are the magnetic and electric quantum numbers. Monodromies arise in this framework in the following way. When we loop with u around the singularities, cycles transform

$$\begin{pmatrix} \beta \\ \alpha \end{pmatrix} \to M \begin{pmatrix} \beta \\ \alpha \end{pmatrix}. \tag{4.11}$$

Obviously this also determines the transformation properties of the periods (4.9). The Picard-Lefshetz formula determines the monodromy action associated with a vanishing cycle ν on any given cycle

$$M_\nu : \quad \gamma \to \gamma + 2(\gamma \circ \nu)\nu. \tag{4.12}$$

Here $\circ$ is the intersection product of one-cycles. It can also be written in terms of a symplectic intersection metric

$$\nu \circ \gamma = \nu^t \Omega \gamma$$

where $\nu = (g, q)$ and $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Geometrically, the intersection number counts the number of intersections and its sign distinguishes whether or not the tangent vectors of the cycles and the normal of the surface form a righthanded system.
In terms of the quantum numbers, we have for the intersection of two vanishing cycles

$$\nu_1 \circ \nu_2 = g_1 q_2 - g_2 q_1 \in \mathbb{Z}. \tag{4.13}$$

The right hand side is in fact the Dirac-Zwanziger condition for the possible quantum numbers. If it vanishes, the dyons are mutually local. Mutually local states can be described by an effective action which treats them both at the same time as fundamental objects.
So the problem now is to find a Riemann surface with the correct monodromies.

4.3 Solution of the Model

The spectral surface for $SU(2)$ can be written in different, equivalent forms, in the sense that different parametrisations of the surface lead to the same physical results. One form given in [10] is

$$y^2 = (x-1)(x+1)(x-u), \tag{4.14}$$

where Λ has been chosen as 1 for simplicity. [3]
The Seiberg-Witten potential λ_{SW} is then

$$\lambda_{SW} = \frac{\sqrt{2}}{2\pi} \frac{\sqrt{x-u}}{\sqrt{x^2-1}} \mathrm{d}x. \tag{4.16}$$

We can choose the branchcuts and cycles to be according to fig. 4.3. The periods

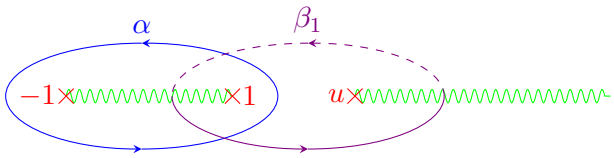

Figure 4.3: Branch cuts (green) and basis cycles - α (blue) and β (violet). The roots of (4.14) are shown in red. The dashed line is on the second sheet.

[3] The convention above is practical for comparisons with $\mathcal{N} = 4$ theories. On the other hand, if we want to add matter, we would have particles of half-integer spin. In this case it is therefore convenient to multiply n_e by 2 and divide a by 2. This leads to a parametrisation [11]

$$y^2 = (x^2 - u)^2 - \Lambda^4 \tag{4.15}$$

which can also be more systematically generalized to higher gauge groups.

are then

$$a_D = \frac{\sqrt{2}}{\pi}\int_1^u \mathrm{d}x \frac{\sqrt{x-u}}{\sqrt{x^2-1}} \qquad a = \frac{\sqrt{2}}{\pi}\int_{-1}^1 \mathrm{d}x \frac{\sqrt{x-u}}{\sqrt{x^2-1}}. \tag{4.17}$$

It is sufficient to check the asymptotic behavior of a and a_D close to $u = \infty$ and $u = 1$.

The behavior of a for $u \to \infty$ can be read immediately from the integral representation

$$a \approx \frac{\sqrt{2}}{\pi}\int_{-1}^1 \mathrm{d}x \frac{\sqrt{u}}{\sqrt{1-x^2}} = \sqrt{2u}.$$

In order to find the behavior of a_D it is useful to switch to the variable $z = x/u$; then the divergent part of the integral is

$$a_D = \frac{\sqrt{2}}{\pi}\int_{1/u}^1 \mathrm{d}z \frac{\sqrt{u}\sqrt{z-1}}{\sqrt{z^2-1/u^2}} \approx \frac{\sqrt{2u}}{\pi}\int_{1/u}^1 \mathrm{d}z \frac{\sqrt{-1}}{z} = i\frac{\sqrt{2u}}{\pi}\log u.$$

For $u \to 1$, perform $z = x/u$ again, and we find

$$\begin{aligned} a_D &= \frac{\sqrt{2}}{\pi}\int_{1/u}^1 \mathrm{d}z \frac{\sqrt{u}\sqrt{z-1}}{\sqrt{z-1/u}\sqrt{z+1/u}} \approx \frac{1}{\pi}\int_{1/u}^1 \mathrm{d}z \frac{\sqrt{z-1}}{\sqrt{z-1/u}} = \\ &= \frac{1}{\pi}\left(\sqrt{z-1}\sqrt{z-\frac{1}{u}} - (1-\frac{1}{u})\log\left(\sqrt{z-1}+\sqrt{z-\frac{1}{u}}\right)\right)\Bigg|_{\frac{1}{u}}^1 = \\ &= \frac{i}{2}(1-\frac{1}{u}) \approx \frac{i}{2}(u-1). \end{aligned}$$

The period a is finite at $u = 1$

$$a = \frac{\sqrt{2}}{\pi}\int_{-1}^1 \frac{\mathrm{d}x}{\sqrt{x+1}} = \frac{4}{\pi}$$

However, the derivative of a with respect to u is divergent. The dominant term can be extracted upon splitting the integration domain into $[-1, s]$ and $[s, 1]$

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}u} &= -\frac{\sqrt{2}}{2\pi}\int_{-1}^1 \frac{\mathrm{d}x}{\sqrt{(x^2-1)(x-u)}} \approx -\frac{\sqrt{2}}{2\pi}\int_s^1 \frac{\mathrm{d}x}{\sqrt{(x+1)(x-1)(x-u)}} \approx \\ &\approx -\frac{1}{2\pi}\int_s^1 \frac{\mathrm{d}x}{x-u} = -\frac{1}{2\pi}\log(x-u)\Big|_s^1 \approx -\frac{1}{2\pi}\log(1-u). \end{aligned}$$

So the expansion of a near $u = 1$ is

$$a = \frac{4}{\pi} - \frac{1}{2\pi}(u-1)\log(1-u).$$

Comparing these expansions with the results obtained in the previous section, we see that they give the correct monodromies.

We can find the formula for the periods using the differential equation approach. The Seiberg-Witten potential satisfies a second order differential equation, the Picard Fuchs equation,

$$4(1-u^2)\frac{\partial^2 \lambda_{SW}}{\partial u^2} = \lambda_{SW}. \tag{4.18}$$

The periods a, a_D both satisfy the same relation. In this case it is fairly easy to derive and check this differential equation. For general gauge groups the situation is more complicated and is better done systematically. A derivation of this procedure can be found in appendix D, the explicit derivation for SU(2) is given there as well. The Picard Fuchs equation, together with the asymptotics, can be used to find an explicit form of the periods.
Equation (4.18) is a Riemann differential equation in the variable $w = \frac{1}{2}(u+1)$

$$w(1-w)\frac{\mathrm{d}^2 f(w)}{\mathrm{d}w^2} - \frac{1}{4}f(w) = 0, \tag{4.19}$$

with parameters $a = b = -\frac{1}{2}$, $c = 0$, as can be seen from (A-2). We are looking for a solution (which will be the period a) such that it is finite for $w = 1$ and for large w it goes as $\sqrt{w}$. A convenient choice is one of the Kummer solutions in terms of the hypergeometric function

$$u_3 = (-w)^{-a} F\left(a, a+1-c, a+1-b; \frac{1}{w}\right).$$

For the period a_D we want a function that vanishes at $w = 1$ and has a logarithm for large w. A good candidate is

$$u_6 = (1-w)^{c-a-b} F(c-a, c-b, c+1-a-b; 1-w).$$

Substituting for the parameters and w, these are

$$u_3 = \left(-\frac{1+u}{2}\right)^{\frac{1}{2}} F\left(-\frac{1}{2}, \frac{1}{2}, 1; \frac{2}{u+1}\right)$$
$$u_6 = \frac{1-u}{2} F\left(\frac{1}{2}, \frac{1}{2}, 2; \frac{1-u}{2}\right).$$

What remains is to check their asymptotics and match them to the asymptotics found from the integral representation; useful relations for this are given in appendix A. For $u \to \infty$ and $u \to 1$, we have

$u \to \infty$	$u \to 1$
$u_3 \approx \pm i\sqrt{\frac{u}{2}}$	$u_3 \approx \pm i\frac{2}{\pi}$
$u_6 \approx -\frac{\sqrt{2}}{\pi}\sqrt{u}(\log u + 3\log 2 - 2)$	$u_6 \approx \frac{1-u}{2},$

from which we see that the periods are

$$a = \mp 2iu_3 = 2\frac{\sqrt{1+u}}{2} F\left(-\frac{1}{2}, \frac{1}{2}, 1; \frac{2}{u+1}\right) \tag{4.20}$$

$$a_D = -iu_6 = i\frac{u-1}{2} F\left(\frac{1}{2}, \frac{1}{2}, 2; \frac{1-u}{2}\right). \tag{4.21}$$

4.4 BPS Spectrum

We shall briefly discuss the stability of BPS states and the BPS spectrum. The spectrum of a theory is the set of all existing BPS states. For a more thorough treatment see [10, 33], for a pedagogical treatment [28].
Any complex number can be regarded as a two-dimensional real vector. It will be useful in the following to regard the central charge as well as the fields a, a_D as two-dimensional vectors. The central charge of a BPS state with given quantum numbers can be regarded as a vector in the complex plane, given by a linear combination of the vectors a_D and a, see (3.6)

$$Z = n_m \alpha_D + n_e a.$$

For a given point u in the moduli space, if the ratio a_D/a is not real (i.e. the vectors a_D, a are not aligned), the numbers a_D and a generate a lattice and all possible central charges are points in this lattice. The possible decays from a state with central charge Z with quantum numbers (n_m, n_e) to a set of final states with central charges Z_i and quantum numbers $(n_{m,i}, n_{e,i})$ are determined by charge conservation

$$Z = \sum_i Z_i.$$

The triangle inequality for the sum of vectors states that

$$|Z| \leq \sum_i |Z_i|. \tag{4.22}$$

Since a decay is impossible if the mass of the original state is lower than the sum of the masses of the resulting particles, decays are possible only if (4.22) is an equality. This case corresponds to all vectors Z_i and Z aligned, i.e. $t_i = Z_i/Z \in \mathbb{R}$ and $\sum t_i = 1$.
If a_D and a are not aligned, this relation requires that all quantum numbers must be proportional to the incoming ones

$$q_i = n_{e,i}/n_e = n_{m,i}/n_m, \quad \forall i.$$

This means that all vectors Z_i are proportional to Z. Furthermore, in order to have a possible decay, the proportionality factors q_i must sum up to 1. In this case the states are only neutrally stable, since the energy of the incoming state is the same as the sum of the final states.
On the other hand, if a_D and a are aligned (i.e. their ratio is real), the lattice of central charges is in fact only a single line. It is now much easier to satisfy the equalities and more states will become unstable. The set of all points u in the moduli space where this happens is called the curve of marginal stability C. It can be determined numerically from the explicit expressions for a_D and a and is shown in figure 4.4. It can also be studied analytically as in [34, 35]. The moduli space is separated by this curve into two distinct regions: the strong and the weak (semi-classical) regions with two different spectra.
The ratio a_D/a takes all values in $[-1, 1]$ as u varies along the curve, with $[-1, 0]$ in

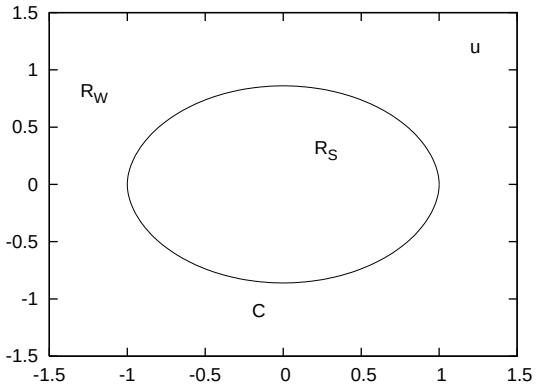

Figure 4.4: *The curve of marginal stability $\mathcal{C}$ separates the moduli space in the strong coupling region R_S and the weak coupling region R_W.*

the upper half plane and $[0,1]$ in the lower half plane. From this it follows immediately that all states with quantum numbers such that $n_e/n_m \in [-1,1]$, become massless somewhere on this curve; more precisely at the point u where $a_D/a = -n_e/n_m$. But the only massless particles with nonzero magnetic charge are the monopole $(1,0)$ and the dyon $(1,1)$, resp., $(1,-1)$. So states that would become massless anywhere else than at $u = \pm 1$ are not part of the spectrum.
Performing a monodromy transformation on a point in the weak coupling region cannot change the theory and thus must leave the spectrum of the weak coupling region S_W invariant. So the spectrum must be invariant under the monodromy transformation of the quantum numbers given in [33]

$$S_W M_\infty = S_W.$$

We know that the spectrum contains at least the monopole and the dyon. Since

$$M_\infty^k = (-1)^k \begin{pmatrix} 1 & -2k \\ 0 & 1 \end{pmatrix},$$

applying this transformation k times to the monopole generates all dyons with even n_e $(1,2k)$ $k \in \mathbb{Z}$ and applying to the dyon all dyons with odd n_e $(1,2k+1)$ $k \in \mathbb{Z}$. The W boson is invariant under M_∞ : $(0,1)M_\infty = -(0,1)$ and so is also part of the weak coupling spectrum. No other dyons can exist: if there were a state (n_m, n_e) with $|n_m| \geq 2$ there would also be all states $(n_m, n_e - 2n_m k)$ and we could find a suitable k such that $(n_e - 2n_m k)/n_m = n_e/n_m - 2k \in [-1,1]$ and this state would become massless. Hence it cannot be present in the spectrum and so the weak coupling spectrum is formed by dyons $(1,n)$ and the W boson $(0,1)$.
The $\mathbb{Z}_2$ symmetry acting on the moduli space requires that the spectra at the points u and $-u$ be the same. This means that for each state (n_m, n_e) at u there must exist a state $(\widetilde{n_m}, \widetilde{n_e})$ at $-u$ such that their masses are equal. Thus there must exist a matrix G which relates the functions a_D and a up to a phase factor

$$\begin{pmatrix} a_D(-u) \\ a(-u) \end{pmatrix} = \mathrm{e}^{i\omega} G \begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \tag{4.23}$$

where ω is a phase. Thus the state (n_m, n_e) at $-u$ corresponds to the state $(n_m, n_e)G$ at u. In the weak coupling region this is just a matter of standard relations among hypergeometric functions, as can be seen from the following. Using the same notation as in the text following (4.19), the change from u to $-u$ corresponds to the change from w to $1-w$. The Kummer solution u_3 can be written either in terms of w or in terms of $1-w$ as

$$\begin{aligned} u_3 &= (-w)^{-a} F\left(a, a+1-c, a+1-b; \frac{1}{w}\right) \\ &= (1-w)^{-a} F\left(a, c-b, a+1-b, \frac{1}{1-w}\right). \end{aligned}$$

So

$$a(-u) = 2\sqrt{\frac{1-u}{2}} F\left(-\frac{1}{2}, \frac{1}{2}, 1; \frac{2}{1-u}\right) = 2u_3$$

and thus $a(-u) = \pm i a(u)$. For a_D we find that

$$a_D(-u) = -iwF(\frac{1}{2}, \frac{1}{2}, 2, w),$$

which is just a multiple of the Kummer solution u_5

$$u_5 = w^{1-c} F(a+1-c, b+1-c, 2-c; z).$$

We can now use a standard relation to change from u_5 to u_3 and u_6 (which are essentially a and a_D)

$$\begin{aligned} \mathrm{e}^{i\pi(1-a)} \frac{\Gamma(1-a)\Gamma(c-b)}{\Gamma(c+1-a-b)} u_6 = \frac{\Gamma(1-a)\Gamma(a+1-c)}{\Gamma(2-c)} u_5 + \\ + \mathrm{e}^{i\pi(1-a)} \frac{\Gamma(c-b)\Gamma(a+1-c)}{\Gamma(a+1-b)} u_3. \end{aligned}$$

In our case this gives

$$u_5 = -i(u_6 - 2u_3)$$

and thus

$$a_D(-u) = -i a_D(u) + i a(u).$$

This holds for $\mathcal{I}m\, u > 0$, in the lower half plane the signs in the phases change as well as the sign between a and u_3. So the matrix G which relates the quantum numbers of equivalent states is in the weak coupling region (for $\mathcal{I}m\, u > 0$)

$$G_W = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \tag{4.24}$$

and the phase ω is $-\frac{\pi}{2}$.

Now we will do a similar analysis for the strong coupling region. In the strong coupling region there is a cut running from $u = -1$ to $u = 1$. We must therefore use analytic continuations of the functions a_D, a which are related to the usual functions

by a monodromy around $u = 1$ (given by (4.5)). Combining both we see that the spectrum in the strong coupling region S_S must be invariant under (in the upper half plane)

$$G_S = M_1^{-1} G_W = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}. \tag{4.25}$$

But now the square of G_S is the negative unit matrix $G_S^2 = -\mathbf{1}$ and thus all states come in $\mathbb{Z}_2$ doublets

$$(n_m, n_e) \in S_S \Longleftrightarrow (n_m + 2n_e, -n_m - n_e) \in S_W,$$

The monopole (1,0) and the dyon (1,-1) form such a doublet. In fact this is the only doublet in the spectrum. For either is $n_e/n_m \in [-1, 0]$ and becomes massless somewhere or its partner becomes massless since then $(n - n_m - n_e)/(n_m + 2n_e) = -(n_e/n_m+1)/(2n_e/n_m+1) \in [-1, 0]$. Since the only massless states are the monopole and the dyon no other states can exist. The argument for the lower half plane is analogous. Thus the strong coupling spectrum contains only the monopole and the dyon (1,-1) (resp., (1,1)).

4.5 SU(2) Quantum Monopoles

We can now combine the t'Hooft-Polyakov monopole and the explicit moduli dependence of the scalar field and use this to study the spatial dependence of the moduli as well as the quantum corrections that arise. We will first review the results of [30]. In the previous chapter the superpotential $\mathcal{F}$ was found, at least implicitly. From the general reduction of the $\mathcal{N} = 2$ action to $\mathcal{N} = 0$ fields given in (2.10)(2.11), (2.12) we can find the action for the physical bosonic fields

$$\mathcal{S}_{\mathcal{F}} = -\frac{1}{4\pi}\mathcal{I}m \int \mathrm{d}^4x\, \mathcal{F}_{AB} \left[\frac{1}{2}(B_i^A + iE_i^A)(B_i^B + iE_i^B) + \nabla_\mu \phi^A \nabla^\mu \bar{\phi}^B + \right.$$
$$\left. + \frac{1}{2}[\phi, \bar{\phi}]^A [\phi, \bar{\phi}]^B \right].$$

As before, we need to find the Hamiltonian. The conjugate momenta receive quantum corrections

$$\Pi_{iA} = -\frac{1}{4\pi}\mathcal{R}e\, \left(\mathcal{F}_{AB}(B_i^B + iE_i^B)\right) \tag{4.26}$$

$$\Pi_A = -\frac{1}{4p}\mathcal{I}m\, (\mathcal{F}_{AB})\nabla_0 \bar{\phi}^B. \tag{4.27}$$

The Hamiltonian is then

$$H = \frac{1}{8\pi}\mathcal{I}m \int \mathrm{d}^3x \mathcal{F}_{AB}(B_i^A B_i^B + E_i^A E_i^B + 2\nabla_i \phi^A \nabla_i \bar{\phi}^B + 2\nabla_0 \phi^A \nabla_0 \bar{\phi}^B +$$
$$+ [\phi, \bar{\phi}]^A [\phi, \bar{\phi}]^B).$$

The Bianchi identities do not receive any quantum corrections, since they are of geometric origin, but the Gauss constraint is now modified

$$\nabla_i \Pi_{iA} + \frac{1}{4\pi} \mathcal{I}m\,(\mathcal{F}_{CD}) f^C_{AB} (\phi^B \nabla_0 \bar{\phi}^D + \bar{\phi}^B \nabla_0 \phi^D) = 0.$$

As before, we shall consider only static configurations and choose the gauge $\nabla_0 \phi^A = 0$. The vacuum is given by the same equations as in the classical case (3.4). The Hamiltonian can again be split into two terms:

$$H_1 = \frac{1}{8\pi} \int \mathrm{d}^3x\; \mathcal{I}m\,(\mathcal{F}_{AB}) \left([\phi, \bar{\phi}]^A [\phi, \bar{\phi}]^B + \right.$$
$$\left. (B_i^A + iE_i^A + \sqrt{2}\mathrm{e}^{i\alpha} \nabla_i \phi^A)(B_i^B - iE_i^B + \sqrt{2}\mathrm{e}^{-i\alpha} \nabla_i \bar{\phi}^B) \right)$$

and a surface term

$$H_2 = -\sqrt{2}\mathcal{I}m \int \mathrm{d}\vec{S} \mathrm{e}^{i\alpha} \left(\frac{1}{4\pi} \vec{B}^A \mathcal{F}_A + \vec{\Pi}_A \phi^A \right)$$

Thus the first term is positive definite as long as the imaginary part of $\mathcal{F}_{AB}$ is positive. The BPS equations for the general monopole are the same as for the classical theory

$$B_j^A + iE_j^A + \mathrm{e}^{i\alpha} \sqrt{2} \nabla_j \phi^A = 0, \tag{4.28}$$

where $\mathrm{e}^{i\alpha}$ is a constant phase. Since the prepontial $\mathcal{F}$ must be gauge invariant, it can depend only on $\phi = \sqrt{\phi^A \phi^A}$. Thus for the derivatives we have

$$\mathcal{F}_A = \mathcal{F}' \frac{\phi^A}{\phi} \tag{4.29}$$

$$\mathcal{F}_{AB} = \mathcal{F}'' \frac{\phi^A \phi^B}{\phi^2} + \frac{\mathcal{F}'}{\phi} \left(\delta_{AB} - \frac{\phi^A \phi^B}{\phi^2} \right). \tag{4.30}$$

given by (4.20) and (4.21).
As in the classical case, we restrict ourselves to the radial ansatz (3.7) :

$$\phi^A = \mathrm{e}^A \phi(r) \qquad A_i^A = \epsilon^A{}_{ij} \mathrm{e}^j \left(\frac{1 - L(r)}{r} \right) \qquad A_0^A = \mathrm{e}^A b(r) \tag{4.31}$$

with electric and magnetic fields

$$B_i^A = \mathrm{e}_i \mathrm{e}^A \frac{L^2 - 1}{r^2} + \mathcal{P}_i^A \frac{L_{,r}}{r} \tag{4.32}$$

$$E_i^A = -\mathrm{e}_i \mathrm{e}^A b_{,r} - \mathcal{P}_i^A \frac{bL}{r} \tag{4.33}$$

and the projector given in (3.10). The derivatives of the prepotential are the dual field $\mathcal{F}' = \phi_D$ and the complex coupling $\mathcal{F}'' = \tau = \mathrm{d}\phi_D / \mathrm{d}\phi$. Thus the derivatives can be written as

$$\mathcal{F}_A = \phi_D \mathrm{e}^A$$
$$\mathcal{F}_{AB} = \tau \mathrm{e}^A \mathrm{e}^B + \frac{\phi_D}{\phi} \left(\delta_{AB} - \mathrm{e}^A \mathrm{e}^B \right).$$

The asymptotic values of the scalar field and the dual field are the solutions of the Seiberg and Witten model given by (4.20) and (4.21).

$$\lim_{r\to\infty}\phi(r)=a \qquad\qquad \lim_{r\to\infty}\phi_D(r)=a_D.$$

These functions are given in terms of hypergeometric functions in (4.20) and (4.21). The hypergeometric function of the arguments $\pm\frac{1}{2}$ can be written in terms of complete elliptic integrals of the first and second kind (A-11). Using (A-8) to lower the value of the third parameter and (A-9) to shift the value of the argument we can rewrite the scalar field and its dual as

$$a(u)=\frac{4}{\pi q}\mathrm{E}(q) \qquad\qquad a_D(u)=-i\frac{4}{\pi q}\left(\mathrm{E}(q')-\mathrm{K}(q')\right) \tag{4.34}$$

where

$$q=\sqrt{\frac{2}{u+1}}$$

and $q'=\sqrt{1-q^2}$ is the complementary modulus. The coupling τ is given by the derivative $\mathrm{d}\alpha_D/\mathrm{d}a$, which is in terms of elliptic integrals

$$\tau=i\frac{\mathrm{K}(q')}{\mathrm{K}(q)}. \tag{4.35}$$

Here τ is the quantum corrected generalized coupling. The imaginary part of τ plays the role of the coupling for the Abelian fields, the imaginary part of the ratio ϕ_D/ϕ plays the role of the coupling for the non-Abelian fields.
The electric and magnetic quantum numbers are defined analogously as before

$$n_m a_D=-\frac{1}{4\pi}\int_{S^2_\infty}\mathrm{d}S_i B_i^A\phi_A^D \qquad\qquad n_e a=-\int_{S^2_\infty}\mathrm{d}S_i\Pi_{iA}\phi^A, \tag{4.36}$$

as well as the central charge $Z=n_m a_D+n_e a$ and the phase $\alpha=\frac{\pi}{2}-\arg Z$.
We can use the ansatz as before and find an explicit expression for the function L

$$L=\frac{\kappa r}{\sinh[\kappa(r+\delta)]}, \tag{4.37}$$

with two integration constants. The only difference is that now we do not require the solution to be finite everywhere, since we cannot expect it to be valid in the strong coupling region. Thus the parameter δ can also be nonzero. The constant κ can be evaluated from the asymptotic value

$$\kappa=\mathcal{R}e\left(\sqrt{2}\mathrm{e}^{i\alpha}a\right) \tag{4.38}$$

The local central charge is defined as $Z=n_m\phi_D+n_e\phi$; we shall show that for the BPS solution it has constant phase. First we shall define two more fields

$$X=\mathcal{R}e\left(\mathrm{e}^{i\alpha}\phi\right) \qquad\qquad X_D=\mathcal{R}e\left(\mathrm{e}^{i\alpha}\phi_D\right)$$

and find a relation between these. Differentiating the BPS equation and taking the real part we have

$$\nabla_i B_i^A + i\nabla_i E_i^A + \mathrm{e}^{i\alpha}\sqrt{2}\nabla_i\nabla_i\phi^A = 0,$$

imposing the Bianchi identity $\nabla_i B_i^A = 0$ and taking the real part we get

$$X_{rr} + 2\frac{X_r}{r} - 2\frac{L^2 X}{r^2}.$$

The dual field ϕ_D satisfies the same equation since inserting the BPS equation in the Gauss constraint $\nabla_i \Pi_{iA} = 0$ gives

$$\nabla_i \Pi_{iA} = \nabla_i \left(\mathcal{R}e\, \mathcal{F}_{AB}\mathrm{e}^{i\alpha}\nabla_i\phi^A\right) = \mathcal{R}e\, \mathrm{e}^{i\alpha}\nabla_i\nabla_i\mathcal{F}_A = 0.$$

At $r \to \infty$ we have by the definition of α that $\mathcal{R}e\,(\mathrm{e}^{i\alpha}Z) = 0$, which implies

$$\lim_{r\to\infty} n_m X_D(r) + n_e X(r) = 0.$$

The covariant derivatives of X and X_D are given by the magnetic field and the momentum. Taking the real part of the BPS equation, we find a relation for $\nabla_i X$, multiplying the BPS equation by $\mathcal{F}_{AB}$ and then taking the real part we find a relation for $\nabla_i X_D$:

$$B_i^A = -\sqrt{2}\nabla_i(\mathrm{e}^A X) \tag{4.39}$$

$$\Pi_{iA} = \frac{1}{4\pi}\sqrt{2}\nabla_i(\mathrm{e}_A X_D). \tag{4.40}$$

From the surface integrals which define the quantum numbers, we can find the asymptotic form of the magnetic field and the conjugate momentum (and thus also of the covariant derivatives)

$$B_i^A - \approx \frac{\mathrm{e}^A \mathrm{e}_i n_m}{r^2}$$
$$\Pi_{iA} - \approx \frac{\mathrm{e}_A \mathrm{e}_i n_e}{4\pi r^2}.$$

Hence, asymptotically

$$4\pi n_m \Pi_{iC} - n_e B_i^A \delta_{AC} = 0 \tag{4.41}$$

which implies $n_m\nabla_i(\mathrm{e}_C X_D) + n_e\nabla_i(\mathrm{e}^A X)\delta_{AC} = 0$. From this we conclude that the relation

$$n_m X_D(r) + n_e X(r) = 0 \tag{4.42}$$

holds for all r. This implies that the phase of the local central charge $Z = n_m\phi_D + n_e\phi$ is constant

$$\mathcal{R}e\,\left(\mathrm{e}^{i\alpha}Z(r)\right) = 0. \tag{4.43}$$

Note that the second order differential equations for X and X_D can be used, together with the Picard-Fuchs equations, to derive a second order differential equation for

$u(r)$.
The curve of marginal stability is given by $\mathcal{I}m\,\frac{\phi_D}{\phi} = 0$. This can be rewritten using the central charge Z and the field X as

$$\mathcal{I}m\,\frac{\phi_D}{\phi}(r) = \frac{1}{|\phi|^2}\mathcal{I}m\,(\bar{\phi}\phi_D) = \frac{1}{n_m|\phi|^2}\mathcal{I}m\,\left((n_m\phi_D + n_e\phi)\bar{\phi}\right) = \\ = \frac{1}{n_m|\phi|^2}\mathcal{R}e\,(\iota\bar{Z}\phi) = \frac{1}{n_m|\phi|^2}|Z(r)|X(r).$$

Thus if $\mathcal{I}m\,\frac{\phi_D}{\phi} = 0$ at a critical radius r_0, this corresponds to two possibilities: either $|Z(r_0)| = 0$ (a solution called Z-pole) or $X(r_0) = 0$ (a solution called X-pole). Note that from the BPS equation we find the explicit form

$$X = -\frac{1}{\sqrt{2}}\frac{\mathrm{d}}{\mathrm{d}r}\ln L = \frac{1}{\sqrt{2}}\left(-\frac{1}{r} + \frac{\kappa}{\tanh[\kappa(r+\delta)]}\right). \tag{4.44}$$

4.6 SU(n) Quantum Monopoles

The idea in 4.2 to describe a theory using an elliptic curve can be generalized to higher gauge groups [23–27, 36]. For SU(n) the equation of the curve is

$$y^2 = p(x)^2 - \Lambda^{2n}, \tag{4.45}$$

$$p(x) = \det(x - <\phi>) = x^n - \sum_{l=0}^{n-2} u_l(<\phi>)x^{n-l} \tag{4.46}$$

the u_k are Weyl invariant Casimir variables. The classical limit can be obtained by $\Lambda \to 0$; this shows that the classical curve $p(x)$ splits into two copies $p(x) \pm \Lambda^n$. Thus the roots of this polynomial also split

$$e_i(u_k) \to e_i^{\pm}(u_k) = e_i(u_2, \ldots, u_n \pm \Lambda^n).$$

The genus of this hyperelliptic curve is $g = n - 1$. There are $n - 1$ holomorphic differentials

$$\omega_{n-i} = \frac{x^{i-1}\mathrm{d}x}{y}, i = 1, \ldots, g \tag{4.47}$$

The one-cycles γ_i, $i = 1, \ldots, 2g$ to create a $(g, 2g)$ dimensional period matrix

$$\Pi_{ij} = \int_{\gamma_j} \omega_i. \tag{4.48}$$

In a symplectic basis of the one-cycles defined by $\alpha_i = \gamma_i$, $\beta_i = \gamma_{g+i}$, $i = 1, \ldots, g$ which satisfies $\alpha_i \circ \beta_j = \delta_{ij}$, $\alpha_i \circ \alpha_j = 0$ and $\beta_i \circ \beta_j = 0$, we can write $\Pi = (A, B$ which gives as the metric on the quantum moduli space $\tau = A^{-1}B$ and by Riemann's second relation $\mathcal{I}m\,\tau$ is positive. This basis can be easily obtained by sorting the e_{2n} roots of 4.45 and defining n one-cycles γ_i which encircle the root pair e_{2i-1}, e_{2i}. These are not independent, but satisfy $\sum_{i=1}^{n}\gamma_i = 0$. The $n - 1$ cycles β_i can be

defined to encircle the pair e_{2i}, e_{2i+1}. The α_i which form a symplectic basis together with β_i can be written as $\alpha_i = \sum_{j \leq i} \gamma_j$. Then

$$A_{ij} = \int_{\alpha_j} \omega_i = \frac{\partial}{\partial u_{i+1}} \phi_j$$
$$B_{ij} = \int_{\beta_j} \omega_i = \frac{\partial}{\partial u_{i+1}} \phi_{D,j}. \tag{4.49}$$

The period matrix τ is defined as $\tau = A^{-1}B$ and equals also the matrix of second derivatives of the prepotential. It forms the metric on the moduli space and is thus positive definite. The $\phi_{D,i}$ and ϕ_i can be expressed as integrals over the Abelian differential of the second kind λ:

$$\phi_{D,i} = \int_{\beta_i} \lambda \qquad\qquad \phi_i = \int_{\alpha_i} \lambda, \tag{4.50}$$

with

$$\lambda = C \frac{1}{2\pi i} \frac{\mathrm{d}}{\mathrm{d}x} p(x) \frac{x \mathrm{d}x}{y}.$$

The constant C is chosen to fit the classical limit.
As shown in [37], one can derive a set of differential equations for the period integrals in a standard way. The procedure and its derivation are quite lengthy; they can be found in appendix D. If two roots become degenerate, some cycle or combination of cycles degenerates, which in turn corresponds to particles becoming massless. Mathematically the points where two roots of a polynomial degenerate can be described using the discriminant, which vanishes at these points

$$\Delta = \prod_{i<j}^{n} (e_i - e_j)^2.$$

Thus the discriminant describes singular curves in moduli space, where the description using the hyperelliptic curve breaks down, and where particles become massless. In the classical case the singular locus where the discriminant vanishes is given as a solution of a complex equation, i.e. as a complex codimension one surface. In the quantum picture the discriminant can be written as a product of two discriminants so the singular locus where the discriminant vanishes is the solution of either one of the complex equations. This means that it is a two-component codimension one surface.

4.7 SU(3) Quantum Monopoles

For SU(3) we have a genus 2 curve given by

$$y^2 = (x^3 - ux - v)^2 - \Lambda^6, \tag{4.51}$$

where u,v are the Casimir variables u_2, u_3, resp. The discriminant is classically

$$\Delta_0 = 4u^3 - 27v^2$$

and for the quantum case it splits as mentioned into two shifted copies

$$\Delta = 64\Lambda^{18}\Delta_+\Delta_- = 64\Lambda^{18}(4u^3 - 27(v-\Lambda^3)^2)(4u^3 - 27(v+\Lambda^3)^2).$$

Thus there are two singular curves where the discriminant becomes zero and states become massless

$$Q_+ : 4u^3 - 27(v-\Lambda^3)^2 = 0 \qquad Q_- : 4u^3 - 27(v-\Lambda^3)^2 = 0. \tag{4.52}$$

These are shown schematically for real u and v in fig. 4.5. In [25] two different

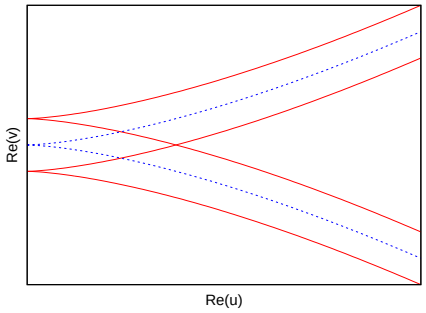

Figure 4.5: The singular curve for classical SU(3) (blue) splits into two separate curves for quantum SU(3) (red).

types of points on the singular curve are pointed out. First the '$\mathbb{Z}_2$ vacua' which are the intersection points of $\Delta_\pm$. These are the three points $u = (\frac{27}{4}\Lambda)^{1/3}\exp i2\pi/3k$, $k = 0,1,2$, $v = 0$. At these points a '$\mathbb{Z}_2$' symmetry remains unbroken, which corresponds to two mutually local dyons becoming simultaneously massless. The other important cases are the '$\mathbb{Z}_3$ vacua' singular points where the derivative of the curve with respect to the moduli becomes zero. These occur at $u = 0$, $v = \mp\Lambda^3$. At these points, also called Douglas-Argyres points, two mutually nonlocal dyons become massless. These points are very interesting since there is no effective action which would describe both dyons simultaneously as local fields. The opportunity to find out more about nonlocal dyons was one of the motivations for further work. The period integrals for SU(3) monopoles were derived in [24], The branch cuts and basis cycles α_i, β_i are chosen as in fig. 4.6. The fields a_i and their duals $a_{D,j}$ are

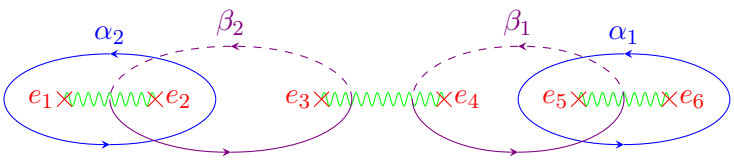

Figure 4.6: Branch cuts (green) and cycles for SU(3) theory: type α - blue, type β - violet. The roots of (4.51) are shown in red. The dashed lines are on the second sheet.

integrals over the Abelian differential of the second kind λ, (4.50)

$$\lambda = \frac{1}{2\pi i}\frac{x(3x^2-u)\mathrm{d}x}{y}. \tag{4.53}$$

In appendix D two differential equations for this potential are derived. The Picard-Fuchs equations can be written in form of two differential operators $\mathcal{L}_1$, $\mathcal{L}_2$, such that all ϕ_j and $\phi_{D,j}$ are solutions of the system $\mathcal{L}_j f = 0$, where f is any of the functions ϕ_i, ϕ_{Dj}. The differential operators turn out to be

$$\begin{aligned}\mathcal{L}_1 &= (27\Lambda^6 - 4u^3 - 27v^2)\partial_u^2 - 12u^2 v\partial_u\partial_v - 3uv\partial_v - u \\ \mathcal{L}_2 &= (27\Lambda^6 - 4u^3 - 27v^2)\partial_v^2 - 36uv\partial_u\partial_v - 9v\partial_v - 3\end{aligned} \tag{4.54}$$

and an additional constraint, which will turn out useful in the solution, arises from the linear combination $u\mathcal{L}_2 - 3\mathcal{L}_1$

$$u\partial_v^2 - 3\partial_u^2 = 0. \tag{4.55}$$

This differential constraint must be satisfied when acting on any ϕ_j and ϕ_{Dj}. The forms of ϕ_i, ϕ_{Di} can be found by combining this with the asymptotic expansions of the integrals. More details of this calculation are given in appendix E. The explicit formulae are given in (E-14).

Chapter 5

Analysis of SU(2) Quantum Monopoles

We showed in the previous chapter that for BPS monopoles both the spatial dependence of the scalar field $\phi(\vec{r})$, e.g. [30], and the dependence on the moduli $\phi(u)$, e.g. [10], are well described. We can combine these to find the spatial dependence of the moduli $u(\vec{r})$. First we derive a first order differential equation for the function $u(r)$ and determine general properties of its solutions. Next we study different types of solution, as well as their properties (energy density, electric and magnetic fields). Most of the work was published in [38] .

5.1 First Order Differential Equation

It is possible to derive a first order differential equation for the moduli $u(r)$. In fact we know that $u(r)$ follows lines of constant Z phase. Now we will show how its size varies.
We shall start by inserting the radial ansatz in the expression for Π_{iA} (4.26) and using the fact that $\mathcal{P}^{AB} = \delta^{AB} - \mathrm{e}^A\mathrm{e}^B$ and $\mathrm{e}_i\mathrm{e}^B$ are orthogonal projections, we have

$$\Pi_{iA} = -\frac{1}{4\pi}\mathcal{R}e\left[\mathcal{P}_{Ai}\left(\frac{L_r}{r} - i\frac{bL}{r}\right)\frac{\phi_D}{\phi} + \tau\,\mathrm{e}_i\mathrm{e}_A\left(\frac{L^2-1}{r^2} - ib_r\right)\right].$$

The functions L, b are real, so we can write everything in terms of real and imaginary parts and obtain

$$\Pi_{iA} = -\frac{1}{4\pi}\left[\mathcal{P}_{Ai}\left(\frac{L_r}{r}\mathcal{R}e\,\frac{\phi_D}{\phi} + \frac{bL}{r}\mathcal{I}m\,\frac{\phi_D}{\phi}\right) + \right. \tag{5.1}$$

$$\left. +\mathrm{e}_A\mathrm{e}_i\left(\frac{L^2-1}{r^2}\mathcal{R}e\;\tau + b_r\mathcal{I}m\;\tau\right)\right]. \tag{5.2}$$

Comparing this with the relation between the magnetic field and the momentum, which was derived in the previous section (4.41) we are left with

$$\frac{n_e}{n_m}\frac{1}{4\pi}\left(\mathrm{e}_i\mathrm{e}_A\frac{L^2-1}{r^2}+\mathcal{P}_{iA}\frac{L_r}{r}\right)=$$
$$=-\frac{1}{4\pi}\left(\mathcal{P}_{iA}\left(\frac{L_r}{r}\mathcal{R}e\,\frac{\phi_D}{\phi}+\frac{bL}{r}\mathcal{I}m\,\frac{\phi_D}{\phi}\right)+\mathrm{e}_A\mathrm{e}_i\left(\frac{L^2-1}{r^2}\mathcal{R}e\;\tau+b_r\mathcal{I}m\;\tau\right)\right).$$

Separating this with respect to the projector $\mathcal{P}_i^A$ and its complement $\mathrm{e}_i\mathrm{e}^A$ we find the following relations

$$\frac{n_e}{n_m}\frac{L^2-1}{r^2} = -\left(\frac{L^2-1}{r^2}\mathcal{R}e\;\tau+b_r\mathcal{I}m\;\tau\right) \tag{5.3}$$
$$\frac{n_e}{n_m}\frac{L_r}{r} = -\left(\frac{L_r}{r}\mathcal{R}e\,\frac{\phi_D}{\phi}+\frac{bL}{r}\mathcal{I}m\,\frac{\phi_D}{\phi}\right). \tag{5.4}$$

Thus although we do not have the explicit dependence of b on r or u we know that

$$b = -\frac{\frac{n_e}{n_m}+\mathcal{R}e\,\frac{\phi_D}{\phi}}{\mathcal{I}m\,\frac{\phi_D}{\phi}}\frac{L_r}{L} \tag{5.5}$$
$$b_r = -\frac{L^2-1}{r^2}\frac{\frac{n_e}{n_m}+\mathcal{R}e\,\tau}{\mathcal{I}m\,\tau} \tag{5.6}$$

When we substitute for b_r in the BPS equation (3.11) we find

$$\sqrt{2}\,\mathrm{e}^{i\alpha}\phi_r=\frac{1-L^2}{r^2}\frac{i}{\mathcal{I}m\,\tau}\left(\frac{n_e}{n_m}+\overline{\tau}\right).$$

But since $\phi(r)=\phi(u(r))$, we can use the chain rule $\phi_r=\phi_u u_r$ and the dependence $\phi(u)$ in (4.34), which gives

$$\phi_u=\frac{q}{\pi}\mathrm{K}(q)\,.$$

Inserting everything in the BPS equation we get

$$\sqrt{2}\,\mathrm{e}^{i\alpha}\sqrt{\frac{2}{1+u}}\frac{\mathrm{K}(q)}{\pi}u_r=\frac{1-L^2}{r^2}\frac{i}{\mathcal{I}m\,\tau}\left(\frac{n_e}{n_m}+\overline{\tau}\right)$$

and we find the differential equation

$$u_r=\frac{\pi}{2}\sqrt{1+u}\frac{1-L^2}{r^2}\frac{\mathrm{e}^{-i\alpha}}{\mathrm{K}(q)}\frac{i}{\mathcal{I}m\,\tau}\left(\frac{n_e}{n_m}+\overline{\tau}\right). \tag{5.7}$$

This is a first order differential equation; the solutions have one integration constant. We will choose it to be the parameter which labels the vacua, i.e. the value of u at infinity $u(r\to\infty)=u_0$. From u_0 the constants α and κ are determined

$$\alpha=\frac{\pi}{2}-\arg(n_m\phi_D(u_0)+n_e\phi(u_0)) \tag{5.8}$$
$$\kappa=\frac{\sqrt{2}n_m|\phi(u_0)|^2}{|n_m\phi_D(u_0)+n_e\phi(u_0)|}\mathcal{I}m\,\frac{\phi_D(u_0)}{\phi_(u_0)}. \tag{5.9}$$

5.2 The Spatial Dependence

The dependence of the solution $u(r)$ on δ is hidden only in the function L and can be removed by changing the parameter from r to X given by (4.44).This changes the differential equation to

$$u_X = \frac{\pi}{2}\sqrt{\frac{1+u}{2}}\frac{\mathrm{e}^{-i\alpha}}{\mathrm{K}(q)}\frac{i}{\mathcal{I}m\,\tau}\left(\frac{n_e}{n_m}+\overline{\tau}\right). \tag{5.10}$$

So δ does not affect the shape of the curve $u(r)$, only its parametrisation. The parameter κ determines the "speed" of the movement.

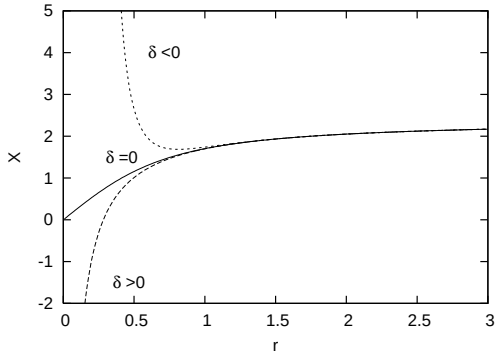

Figure 5.1: The dependence X(r)

We see from fig. 5.2 that there are essentially three different cases depending on whether δ is positive, negative or zero. When $\delta > 0$ X changes monotonically from $X = \kappa/\sqrt{2}$ at $r = \infty$ to $X \to -\infty$ at $r = 0$. For $\delta = 0$, X also changes monotonically but ends at $X = 0$ for $r = 0$. The $\delta < 0$ case is quite different. For $r = \infty$, it starts at $X = \kappa/\sqrt{2}$ and decreases. For finite r, there is a minimum and then X starts to increase and goes to ∞ at $r = -\delta > 0$. The minimum of X is at an r which is a solution of the condition

$$\kappa r = \sinh[\kappa(r+\delta)]., \tag{5.11}$$

In the limit $\delta \to \infty$ the r dependence of X is very simple $X = 1/\sqrt{2}(\kappa - 1/r)$. As we will see later, in this limit, all non-Abelian parts of the fields are suppressed.

5.3 The Attractor Equation

Using X as a parameter and using the Seiberg-Witten metric $\mathrm{d}s^2 = \mathcal{I}m\,\tau \mathrm{d}\phi \mathrm{d}\bar{\phi}$, the equation for u becomes

$$u_X = \frac{ie^{-i\alpha}}{2n_m} g^{u\bar{u}}\partial_{\bar{u}}\bar{Z} = -\frac{1}{n_m} g^{u\bar{u}}\partial_{\bar{u}}|Z| \tag{5.12}$$

where we also have used the local central charge $Z(u) = n_m\phi_D(u) + n_e\phi(u)$ to rewrite the equation in a suggestive form. In fact, since there is a one to one map between

u and Z we may use Z as a coordinate instead of u. This leads us to the equation

$$\frac{\mathrm{d}Z}{\mathrm{d}X} = \frac{ie^{-i\alpha}}{2n_m} g^{u\bar{u}} \partial_u Z \partial_{\bar{u}} \bar{Z} = \frac{ie^{-i\alpha}}{2n_m} g^{Z\bar{Z}}, \tag{5.13}$$

where $g_{Z\bar{Z}}$ is the Seiberg-Witten metric in Z coordinates, or equivalently or

$$\frac{\mathrm{d}|Z|}{\mathrm{d}X} = \frac{1}{2n_m} g^{u\bar{u}} \partial_u |Z| . \partial_{\bar{u}} |Z| \tag{5.14}$$

This is an attractor equation as first discovered in [39] and it can alternatively be derived taking the zero gravity limit of the ordinary attractor equations. Using this form of the equation and the fact that the Seiberg-Witten metric is positive definite, we see that $\frac{d\ |Z|}{dX} > 0$. This means that when X is decreasing (which is the usual situation for decreasing r), $|Z|$ will decrease and $|Z| = 0$ is an attractor point.

5.4 General Properties of the Solutions

Using the above relations we may write

$$\frac{\mathrm{d}|Z|}{\mathrm{d}r} = \frac{1}{2n_m} g^{u\bar{u}} \partial_u |Z| \partial_{\bar{u}} |Z| \frac{dX}{dr}, \tag{5.15}$$

as well as

$$\arg \frac{dZ}{dX} = \frac{\pi}{2} - \alpha. \tag{5.16}$$

It follows from this and (5.8) that the phase α and the phase of the central charge sum up to $\frac{\pi}{2}$

$$\alpha = \frac{\pi}{2} - \arg Z. \tag{5.17}$$

From this we can see that the curve $Z(X)$ is a straight line in the Z-plane going from $Z_0 = Z(\infty)$ to $Z = 0$.
From fig. 5.2 we see that the derivative $\frac{dX}{dr}$ is in general positive, so $|Z|$ will decrease when we decrease r. However, if the sign of the derivative $\frac{dX}{dr}$ changes (which is the case for $\delta < 0$), the phase of the derivative jumps by π and $|Z|$ starts to increase for decreasing r, ending up at $X = \infty$ for $r = -\delta$. This behavior, that $|Z|$ "bounces" at some value of r and starts to increase leads us to call this class of solutions, bouncing solutions.

The point at which the bouncing solution turns around is given by (5.11). Whether the solution hits first the origin $Z = 0$ or the curve of marginal stability elsewhere distinguishes the X- and Z-poles. Since X is given by (4.44), an X-pole corresponds to the condition for the critical radius

$$\tanh[\kappa(r + \delta)] = \kappa r, . \tag{5.18}$$

Physical, i.e. positive solutions $r > 0$, occur only for positive δ. As a consequence we see that the bouncing solutions always lie outside the curve of marginal stability. In fig. 5.2 the curve of marginal stability is shown as well as some solutions. We can see that if $\alpha(u_0) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the solution is a Z-pole; otherwise it is an X-pole.

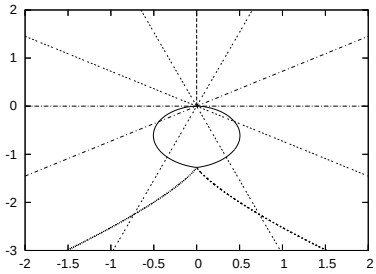

Figure 5.2: *Solutions, the curve of marginal stability and the branch cut in the complex plane of the central charge. Solutions with phase* $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *hit the curve of marginal stability* C *at* $Z = 0$ *and are Z-poles, the others are X-poles or XZ-poles (if* $\alpha = \pm\frac{\pi}{2}$*)*

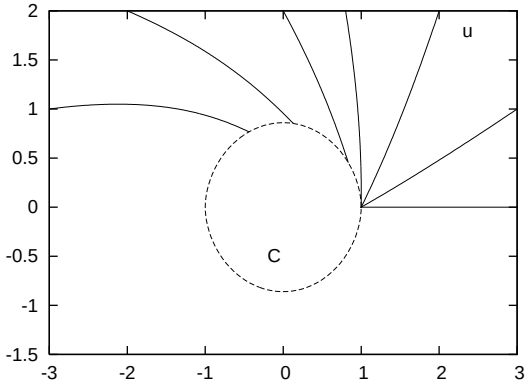

Figure 5.3: *Solutions and the curve of marginal stability in moduli space. The BPS solutions end on the curve of marginal stability, since the BPS equations do not hold inside.*

5.5 Monopole Solution

We concentrate now, unless otherwise stated, on the case with quantum numbers $n_m = 1$, $n_e = 0$. In this case, the central charge coincides with the dual scalar field. Also the solutions to the differential equation are symmetric with respect to the real u-axis, since

$$\left.\frac{\mathrm{d}u}{\mathrm{d}r}\right|_u = \overline{\left.\frac{\mathrm{d}u}{\mathrm{d}r}\right|_{\bar{u}}}.$$

The field ϕ_D is zero at $u = 1$ so the Z-poles end at this point in the u-plane. We would like to see what value the parameter r acquires at this point. At this point $\phi = 4/\pi$ and this corresponds to $X = 4/\pi \cos\alpha$. From the definition of X in (4.44), this corresponds in terms of r to the condition

$$-\frac{1}{r} + \kappa \coth[\kappa(r+\delta)] = \sqrt{2}\cos\alpha \frac{4}{\pi}. \tag{5.19}$$

For X-poles, the factor $\cos\alpha$ is negative and thus X becomes zero before this point is reached. For Z-poles, (i.e. $\alpha \in (-\pi/2, \pi/2)$), the factor $\cos\alpha$ is always positive and thus $u = 1$ is reached before $X = 0$. For positive $\cos\alpha$ and $\delta < 0$, there are in principle two possibilities: either the solution "bounces" back at some point or it reaches $u = 1$. This is governed by the value of δ: for a value of δ greater than a certain δ_0 the solution will reach $Z = 0$ before it reaches the point of the bounce. If δ is smaller than δ_0 the solution will be a bouncing solution. This particular value $\delta_0 < 0$ solves the equation

$$\frac{8\sqrt{2}}{\pi}\frac{\kappa\cos\alpha}{\kappa^2 - \frac{32}{\pi^2}} = \sinh\left[\frac{8\sqrt{2}\cos\alpha}{\pi}\frac{\kappa}{\kappa^2 - \frac{32}{\pi^2}\cos^2\alpha} + \kappa\delta_0\right], \tag{5.20}$$

the value δ_0 and the corresponding solution are called critical. All other cases are called noncritical.

5.6 Electric and Magnetic Fields of Monopoles

We substitute in (4.32) for b and b_r from (5.5) and (5.6) and we find the following expressions for the electric/magnetic fields

$$B_{\mathrm{abel}} = \frac{L^2 - 1}{r^2} \qquad B_{\mathrm{non}} = \frac{L_r}{r} \tag{5.21}$$

$$E_{\mathrm{abel}} = \frac{L^2 - 1}{r^2}\frac{\mathcal{R}e\,\tau + n_e/n_m}{\mathcal{I}m\,\tau} \qquad E_{\mathrm{non}} = \frac{L_r}{r}\frac{\mathcal{R}e\,\frac{\phi_D}{\phi} + \frac{n_e}{n_m}}{\mathcal{I}m\,\frac{\phi_D}{\phi}}. \tag{5.22}$$

In the classical case, for the t'Hooft-Polyakov monopole, the dual field is just a multiple of the scalar $\phi_D = \tau\phi$, with τ constant. So the factors relating the Abelian and the non-Abelian fields are equal and constant. For $n_e = 0$, since the complex coupling is $\tau = \theta/2\pi + 4\pi i/g^2$, the electric fields are related to the magnetic fields in terms of the coupling constant g and the theta angle θ as

$$E = \frac{\theta g^2}{8\pi^2} B. \tag{5.23}$$

This holds for both for Abelian and non-Abelian fields.

Classically, the non-Abelian magnetic field is always nonzero. In the quantum case the non-Abelian magnetic field can become zero only for the X-pole or in the $\delta \to \infty$ limit (see below).

The asymptotic behavior for large r of the electric/magnetic fields is the same for all types of solutions listed below. The Abelian fields have for large r a $1/r^2$ behavior, the non-Abelian fields vanish exponentially

$$B_{\rm abel} \approx -\frac{1}{r^2} \qquad E_{\rm abel} \approx -\frac{1}{r^2}\frac{\mathcal{R}e\,\tau(u_0) + \frac{n_e}{n_m}}{\mathcal{I}m\,\tau(u_0)} \tag{5.24}$$

$$B_{\rm non} \approx \frac{\kappa}{r}{\rm e}^{-\kappa(r+\delta)} \qquad E_{\rm non} \approx \frac{\kappa}{r}{\rm e}^{-\kappa(r+\delta)}\frac{\mathcal{R}e\,\frac{\phi_D}{\phi}(u_0) + \frac{n_e}{n_m}}{\mathcal{I}m\,\frac{\phi_D}{\phi}(u_0)} \tag{5.25}$$

In the quantum case there are several different types of behavior: bouncing solutions, X-poles, Z-poles and solutions which are on the border between two types. In the following we shall briefly describe the various types of behavior. All calculations are straightforward, sometimes rather lengthy. For Z-poles one needs to use expansions of various fields around $u = 1$; these are given in the appendix B.

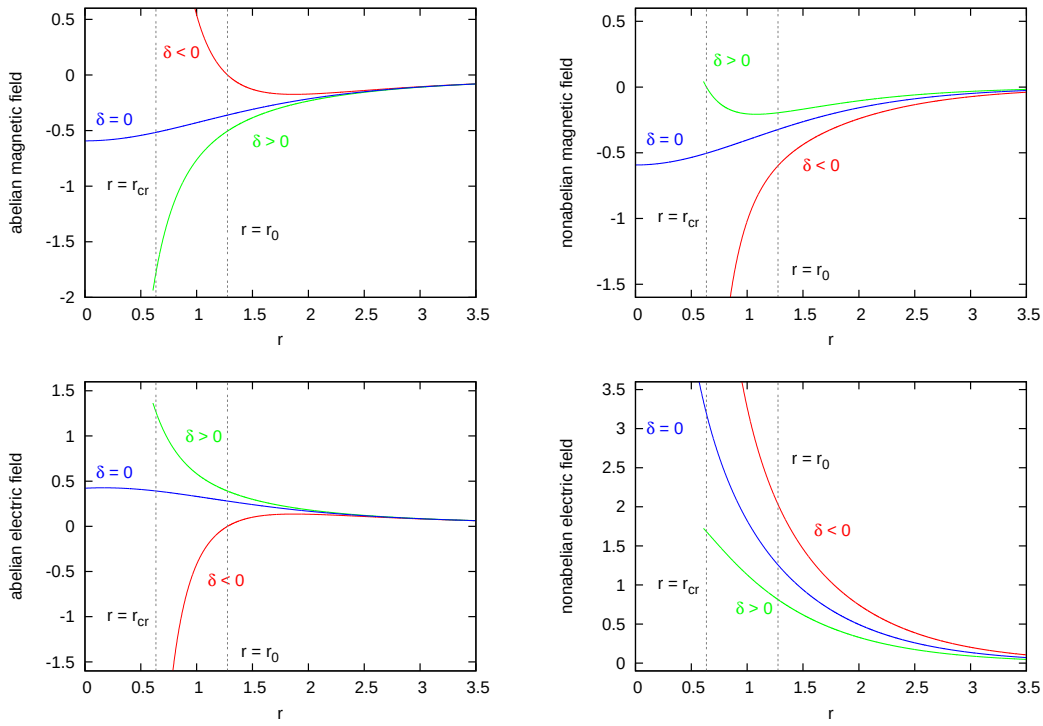

Figure 5.4: The magnetic and electric fields of an X-pole ($\delta > 0$), a bouncing solution ($\delta < 0$), and a critical X-pole ($\delta = 0$).

5.6.1 $\delta \to \infty$ limit

For large δ, the magnetic fields have the following behavior

$$B_{\text{abel}} = -\frac{1}{r^2} + 4\kappa^2 \mathrm{e}^{-\kappa(r+\delta)} \qquad\qquad B_{\text{non}} = \frac{2\kappa}{r}\mathrm{e}^{-\kappa(r+\delta)}.$$

The electric fields include these factors, as given in (5.21), (5.22), but there is also the δ-dependence of the coupling τ, so their behavior is more complicated.
From the above we see that in the $\delta \to \infty$ limit only the Abelian fields survive. Then the Abelian magnetic field is $B_{ab} = 1/r^2$ but the Abelian electric field has a more complicated dependence. Thus we can identify the $\delta \to \infty$ as the Abelian limit, where there are only Abelian fields. In this case the $X(r)$ (4.44) dependence simplifies to $X = 1/\sqrt{2}(\kappa - 1/r)$, so the X-pole hits the curve of marginal stability at $r = 1/\kappa$. The condition (5.19) which describes the point r at which a Z-pole reaches $u = 1$ simplifies also to $r = (\kappa - \sqrt{2}\cos\alpha 4/\pi)^{-1}$.

5.6.2 Bouncing solution

As shown before, a bouncing point occurs for either $\delta < 0$ and $\alpha \notin (-\frac{\pi}{2}, \frac{\pi}{2})$, or for $\delta < \delta_0$ and $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. At the bouncing point both the Abelian magnetic field and the Abelian electric field vanish, since the factor $L^2 - 1$ vanishes. The non-Abelian fields remain finite.

5.6.3 X-pole

The solution is an X-pole for $\delta \geq 0$ and $\alpha \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$. The non-Abelian magnetic field vanishes at the point where the solution $u(r)$ hits the curve of marginal stability. As for the non-Abelian electric field, $\mathcal{I}m\,(\phi_D/\phi)$ also vanishes. A careful calculation shows that both effects cancel and the remaining limit is finite.

The case $\delta = 0$ is on the border between a bouncing solution and an X-pole: the solution hits the curve of marginal stability only at $r = 0$. Both magnetic fields remain finite and acquire the same value $B_{\text{abel}} = B_{\text{non}} = -\kappa^2/3$. The Abelian electric remains finite as well, since the factor $(L^2 - 1)/r^2$ is finite, the non-Abelian electric field however diverges, since the factor L_r/r remains finite but $\mathcal{I}m\,(\phi_D/\phi)$ vanishes.

5.6.4 Z-pole

Z-poles correspond to solutions which have a central charge phase $\arg Z \in [-\pi/2, \pi/2]$ and δ larger than the critical value δ_0. The magnetic fields remain finite upon reaching $u = 1$. As shown in appendix B, the factor $\mathcal{R}e\,\tau/\mathcal{I}m\,\tau$ goes to zero for $u \to 1$, so the Abelian electric field vanishes. It is also shown in the appendix that $\frac{\mathcal{R}e\,(\phi_D/\phi)}{\mathcal{I}m\,(\phi_D//\phi)} \to \tan\alpha$, so the non-Abelian field approaches a finite value.

For the critical δ value $\delta = \delta_0$, the point $u = 1$ is reached exactly at the bouncing point. Thus the factor $L^2 - 1$ becomes zero at this point and causes the Abelian magnetic field to also vanish. This holds also for the Abelian electric field, but it is zero due also to the $\mathcal{R}e\,\tau/\mathcal{I}m\,\tau$ factor. The behavior of the non-Abelian fields does not differ from the noncritical case.

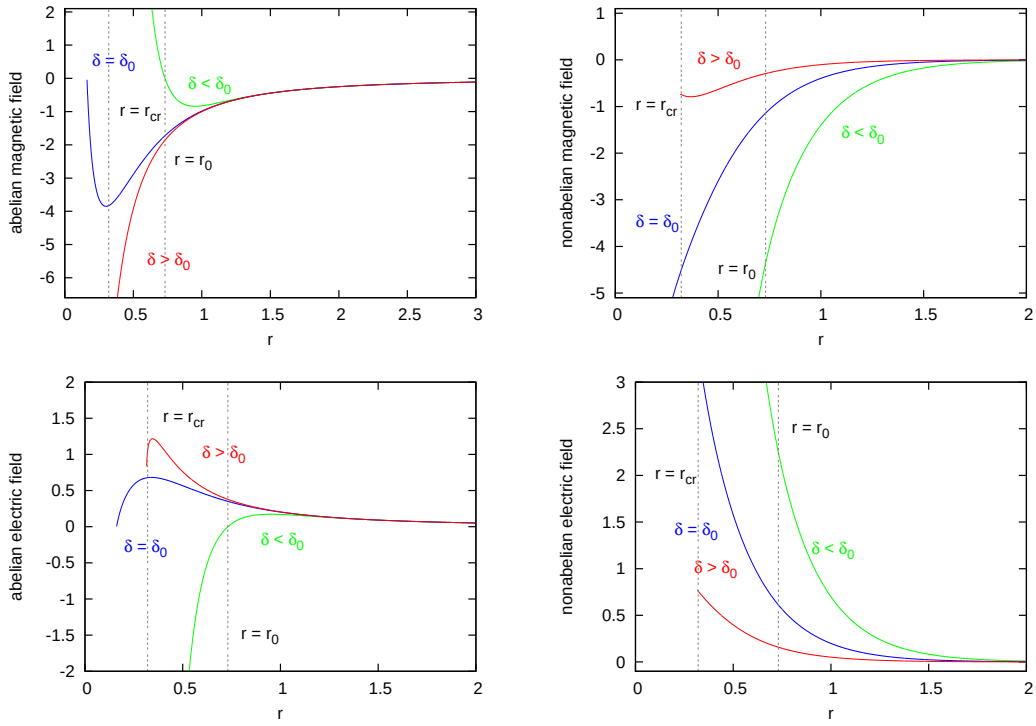

Figure 5.5: The magnetic and electric fields of a Z-pole ($\delta > \delta_0$), a bouncing solution ($\delta < \delta_0$) and a critical Z-pole ($\delta = \delta_0$).

5.6.5 XZ-pole

When $\alpha = \pm\frac{\pi}{2}$ and $\delta > \delta_0 = 0$, the solution is at the same time a Z-pole and an X-pole, i.e. it reaches $u = 1$ when $X = 0$. In this case the magnetic fields are the same as for X-poles, i.e. the Abelian magnetic fields remains finite, the non-Abelian field vanishes. The Abelian electric field vanishes as for a Z-pole. The non-Abelian field must be checked separately, since we cannot use the same limit as before. The dual field ϕ_D (being essentially the central charge) is for $\alpha = \pm\frac{\pi}{2}$ real, so we need to study only the limit $-L_r/r\ \mathcal{R}e\,\phi/\mathcal{I}m\,\phi$. The nontrivial part of this is

$$\lim_{u\to 1}\frac{L_r}{\mathcal{I}m\,\phi} = \lim_{u\to 1}\frac{L_{rr}}{\mathcal{I}m\,(\phi_r)} = \pm\sqrt{2}\,\frac{r^2 L_r r}{1-L^2}\bigg|_{r=r_*},$$

where r_* is the point where $u = 1$ is reached and is given by (5.19). Inserting this and the value $\phi(u = 1) = 4/\pi$ in the expression for the non-Abelian field, we find the value $\pm\sqrt{2}\frac{4}{\pi}\frac{\sqrt{1+\kappa^2 r_*^2}}{r_*}$.

The case $\delta = 0$ must be treated separately. In this case the solution ends at $r = 0$, at $u = 1$ and at the same time $X = 0$. As in the case of an X-pole with zero δ the magnetic fields acquire the same value. The Abelian electric field vanishes as for any Z-pole. We see from the calculation of the non-Abelian electric field of an XZ-pole that it diverges as $1/r$.

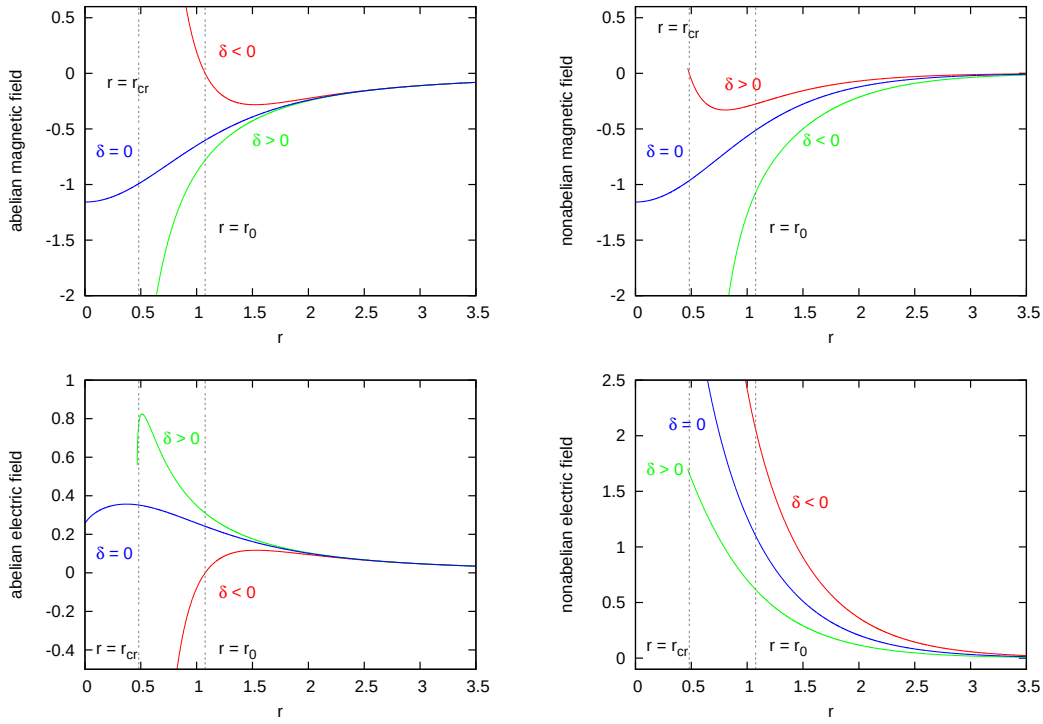

Figure 5.6: The magnetic and electric fields of an XZ-pole ($\delta > 0$), a bouncing solution ($\delta < 0$), and a critical XZ-pole ($\delta = 0$)).

5.7 Energy Density

The energy of a configuration is given by the Hamiltonian

$$H = \frac{1}{8\pi}\mathcal{I}m \int \mathrm{d}^4x\, \mathcal{F}_{AB}(E_i^A E_i^B + B_i^A B_i^B + 2\nabla_i \phi^A \nabla_i \bar{\phi}^B), \tag{5.26}$$

so the energy density is

$$\mathcal{E} = \frac{1}{8\pi}\mathcal{I}m\, \mathcal{F}_{AB}(E_i^A E_i^B + B_i^A B_i^B + 2\nabla_i \phi^A \nabla_i \bar{\phi}^B).$$

For a BPS solution we see that the electromagnetic field and the Higgs field each carry one half of the total energy. We can use the BPS equation to substitute for the Higgs field and regard just the double of the electromagnetic energy. In the radial ansatz, the coupling $\mathcal{F}_{AB}$, the electric and magnetic fields split in Abelian and non-Abelian components. The energy density splits into an Abelian and a non-Abelian part as well, with τ being the Abelian coupling and $\frac{\phi_D}{\phi}$ the non-Abelian coupling

$$\mathcal{E} = \frac{1}{4\pi}\mathcal{I}m\, \tau(B_{\text{abel}}^2 + E_{\text{abel}}^2) + \frac{1}{2\pi}\mathcal{I}m\, \frac{\phi_D}{\phi}(B_{\text{non}}^2 + E_{\text{non}}^2), \tag{5.27}$$

the Abelian and non-Abelian fields are given in eq. (5.21) and (5.22).
The t'Hooft-Polyakov monopole is the classical case with $\mathcal{F} = \frac{1}{2}\tau\phi^A\phi^A$ (and $\delta = 0$). The Abelian and non-Abelian couplings are the same, furthermore this coupling is fixed by its asymptotic value at infinity $\tau(u_0)$. Thus the classical energy is

$$\mathcal{E} = \frac{\mathcal{I}m\,\tau}{4\pi}\left(B_{\text{abel}}^2 + E_{\text{abel}}^2 + 2B_{\text{non}}^2 + 2E_{\text{non}}^2\right). \tag{5.28}$$

The Hamiltonian can be written as a term including the square of the BPS equation H_0 and a total derivative term, which can be rewritten as a surface term

$$\begin{aligned} H &= H_0 - \sqrt{2}\mathcal{I}m\, \mathrm{e}^{i\alpha} \int \mathrm{d}^3x \, \left(\frac{1}{4\pi}\nabla_i(B_A^i \phi_D^A) + \nabla_i(\Pi_A^i \phi^A)\right) = \\ &= H_0 - \sqrt{2}\mathcal{I}m\, \mathrm{e}^{i\alpha} \int_{S^2_\infty} \mathrm{d}^2 S_i \left(\frac{1}{4\pi}B_A^i \phi_D^A + \Pi_A^i \phi^A\right) + \\ &+ \sqrt{2}\mathcal{I}m\, \mathrm{e}^{i\alpha} \int_{S^2_{r_0}} \mathrm{d}^2 S_i \left(\frac{1}{4\pi}B_A^i \phi_D^A + \Pi_A^i \phi^A\right). \end{aligned}$$

According to the definition of the electric and magnetic quantum numbers, the surface term at infinity is equal to $-(n_m a_D + n_e a)$. We can use the relation between the magnetic field and the conjugate momentum

$$4\pi n_m \Pi_i^A - n_e B_i^A = 0$$

to write the third term only in terms of the magnetic field

$$\begin{aligned} H &= H_0 + \sqrt{2}\mathcal{I}m\, \mathrm{e}^{i\alpha}(n_m a_D + n_e a) + \\ &+ \sqrt{2}\mathcal{I}m\, \mathrm{e}^{i\alpha} \int_{S^2_{r_0}} \mathrm{d}^2 S_i B_A^i \frac{1}{4\pi}\left(\phi_D^A + \frac{n_e}{n_m}\phi^A\right). \end{aligned}$$

According to our ansatz, the magnetic field splits into an Abelian and a non-Abelian part $B_i^A = \mathrm{e}^i \mathrm{e}^A (L^2-1)/r^2 + (\delta_i^A - \mathrm{e}^i \mathrm{e}^A) L'/r$ and the scalar fields are pure Abelian $\phi^A = \phi \mathrm{e}^A$. Multiplying these by $\mathrm{d}^2 S^i = \mathrm{d}\Omega r^2 \mathrm{e}^i$ leaves only the Abelian terms

$$\begin{aligned} H &= H_0 + \sqrt{2}\mathcal{I}m\, \mathrm{e}^{i\alpha}(n_m a_D + n_e a) + \\ &+ \sqrt{2}\mathcal{I}m\, \mathrm{e}^{i\alpha} \int_{S^2_{r_0}} \frac{1}{4\pi n_m} \mathrm{d}\Omega (L^2-1)(n_m \phi_D + n_e \phi) = \\ &= H_0 + \sqrt{2}\mathcal{I}m\, \mathrm{e}^{i\alpha}(n_m a_D + n_e a) + \\ &+ \sqrt{2}\mathcal{I}m\, \mathrm{e}^{i\alpha} \frac{1}{n_m}(L^2-1)(n_m \phi_D + n_e \phi)\Big|_{r=r_0} . \end{aligned}$$

The second term includes the asymptotic value of the central charge $Z_0 = Z(r = \infty) = n_m a_D + n_e a$. As for the third term, since the phase of the central charge is constant, we can rewrite it in much the same way as the second term and we get for a BPS state ($H_0 = 0$)

$$H = \sqrt{2}|Z(r=\infty)| + \sqrt{2}\frac{1}{n_m}(L^2(r_0)-1)|Z(r_0)|.$$

The term $L^2 - 1$ is (up to a factor r^2) the reparametrisation term $\mathrm{d}X/\mathrm{d}r$. It is negative for $\delta \geq 0$ with $L \to 1$ for $\delta = 0$ and $r \to 0$. For negative delta, however, it can change sign: from negative (at large r) to positive (at small r). This shows that the third term changes the total energy of the configuration. For Z-poles the energy of the configuration is lowered for any r larger than the critical value r_{cr}, at which the point $u = 1$ (and thus $Z = 0$) is reached. For X-poles it is lowered for all r larger than the value at which the curve of marginal stability is crossed and the BPS equations do not necessarily have to hold any more. For bouncing solutions the energy is lowered for r larger than the bouncing point, it is increased for smaller values and tends to infinity for $r \to -\delta$.

5.8 Weakly Coupled Monopole Solution

The behavior of the energy dependence enables us a to construct a completely weakly coupled monopole solution by utilizing the properties of the bouncing solution. If we choose δ in such a way that the value of u for which the solution turning point is in the region where we may trust the low-energy effective description, we may cut off the solution there. As was shown in the previous sections, the Abelian components of the electric and magnetic field are zero at the turning point and there is no contribution to the energy from the inner boundary. We may therefore combine the bouncing solution and a suitable Higgs vacuum to get a solution which is defined for all values of r. The Higgs vacuum can be chosen as follows: The scalar field ϕ is constant in the radial direction, with the value it has at the cut off point

$$\Phi = \phi(r_0) T_R. \tag{5.29}$$

This is correctly defined everywhere except the origin, where the directional angles ϑ, φ lose their meaning. But we can use a gauge transformation to transform the

field to say the 3 direction $g^{-1}T_R g = T_3$, with g given by

$$g = \begin{pmatrix} \cos\frac{\vartheta}{2} & -\mathrm{e}^{-i\varphi}\sin\frac{\vartheta}{2} \\ \mathrm{e}^{i\varphi}\sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix}. \tag{5.30}$$

The gauge field is chosen to be pure gauge, such that vanishes in the gauge in which the scalar field points in the 3 direction, i.e. $A = -ig^{-1}\mathrm{d}g$,

$$A = T_\Phi \mathrm{d}\vartheta + (T_R - T_3)\mathrm{d}\varphi. \tag{5.31}$$

As before, this form is well defined everywhere except the origin, where we can use its gauge transformed version. As can be easily checked, $\nabla_i \Phi = 0$, and this solution is indeed a Higgs vacuum, see (3.4).

The scalar field, and the Abelian components of the covariant derivative and electromagnetic fields are continuous at the boundary. The non-Abelian electromagnetic fields as well as the non-Abelian part of the covariant derivative of the scalar field, however, jump from a non-zero value outside the boundary to zero inside. The non-Abelian fields are not invariant under the residual $U(1)$, instead, the $U(1)$ rotates the angular Θ and Φ components. The non-Abelian fields do not have any charge associated with them, neither do they contribute to the total energy. They do, however, contribute to the total energy density. This means that there is a shell-like discontinuity at the cutoff radius.

5.9 Other Dyons

Several solutions for the quantum numbers $(1, -1)$ are shown in fig. 5.7. Again, we see the different behavior for Z-poles and X-poles. The dyon $(1, -1)$, resp. $(1, 1)$, becomes massless at $u = -1$. At this point the scalar field ϕ approaches the value $\pm 4/\pi i$; the sign is positive if it approaches $u = -1$ from the $\mathcal{I}m\, u > 0$ halfplane and negative otherwise. Then the value of X at $u = -1$ is $X = \mp 4\sqrt{2}/\pi \sin\alpha$. From this it follows that for a solution which approaches $u = -1$ from the lower halfplane, to be a Z-pole, we must have $\alpha \in (0, \pi)$, corresponding to $\arg Z \in (-\frac{\pi}{2}, \frac{\pi}{2})$. From fig. 5.8, we see that $\arg Z \in (-\frac{\pi}{2}, 0)$ is beyond the cut and thus corresponds to an analytic continuation. We see as well, that indeed solutions starting in the first quadrant hit the curve of marginal stability at $Z = 0$ and are thus Z-poles.

Dyons with a higher electric quantum number are only marginally stable and decay upon crossing the curve of marginal stability. We can see from fig. 5.10, 5.9, that all solutions are X-poles. On the curve of marginal stability it becomes energetically favorable to decay into three $(1, 1)$ dyons and and two $(-1, 0)$ antimonopoles. These then move on towards $u = \pm 1$ where they become massless.

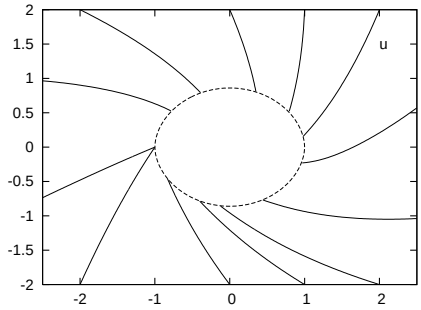

Figure 5.7: *Solutions for* $(n_m, n_e) = (1, -1)$ *and the curve of marginal stability in moduli space. Solutions which end at* $u = 1$ *are Z-poles, the others are X-poles. Z-poles correspond to* u_0 *with both real and imaginary parts negative.*

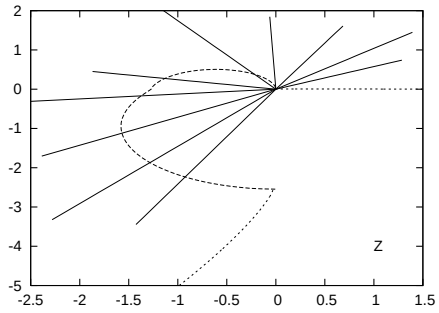

Figure 5.8: *Solutions for* $(n_m, n_e) = (1, -1)$ *and the curve of marginal stability as well as the branch cut in the central charge plane.*

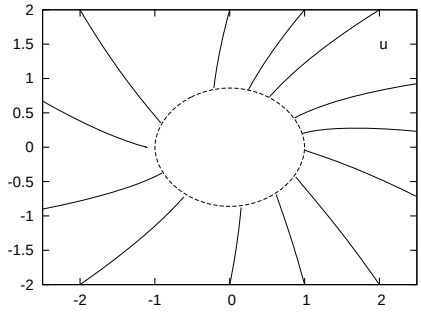

Figure 5.9: *Solutions for* $(n_m, n_e) = (1, 3)$ *and the curve of marginal stability in moduli space. All solutions hit the curve of marginal stability and are therefore X-poles. Upon crossing the curve of marginal stability they decay into (-1,0) antimonopoles and (1,1) dyons.*

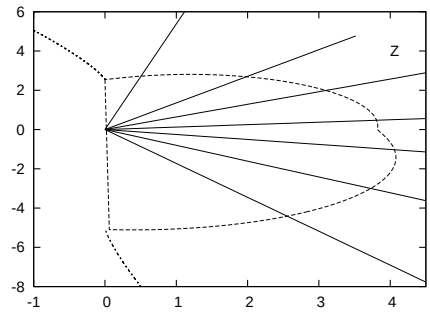

Figure 5.10: *Solutions for* $(n_m, n_e) = (1, 3)$ *and the curve of marginal stability as well as the branch cut in the central charge plane. All solutions hit the curve of marginal stability where they decay into (-1,0) antimonopoles and (1,1)dyons.*

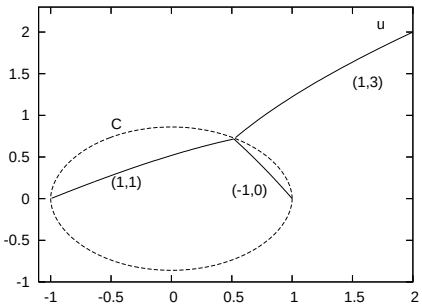

Figure 5.11: *The decay of a* $(1, 3)$ *dyon into* $(-1, 0)$ *antimonopoles and* $(1, 1)$ *dyons, which become massless at* $u = 1$*, resp.,* $u = -1$.

Chapter 6

Analysis of SU(3) Quantum Monopoles

The analysis done in chapter 5 can be done similarly also for SU(3). The situation is more involved for SU(3), since there are two moduli u, v and a more complicated structure of singular points.
Except for the notations and conventions, this chapter is original work, not yet published.
After briefly defining necessary notations and conventions we derive a system of first order differential equations for the spatial dependence of the moduli $u(r)$, $v(r)$. The numerical treatment of the SU(3) period integrals is more complicated than for SU(2) and is commented on. The structure of the curves of marginal stability are elaborated and numerical examples shown. Examples of moduli motion are shown in the end. Since the moduli space is $\mathbb{C}^2$, it is difficult to visualize solutions and curves. Therefore, we focus on those cases where the problem reduces to $\mathbb{C}$.

6.1 Definitions

In SU(3) differential equations for the moduli u, v can also be found. The derivation follows the derivation of the first order equation for SU(2) monopoles in section 5.1. The generalization of most of the formulas given in section 4.5 from SU(2) to SU(3) is straightforward, but for clarity and future reference, a brief derivation will be given below.

We use most two bases in our calculations; both are given explicitly in appendix C. For the classical description, a modification of the radial gauge is the easiest; it uses the standard basis based on the Gell-Mann matrices T_A. On the other hand, the quantum description is easier in a diagonal gauge with a more symmetric basis of the Cartan subalgebra $\mathcal{T}_i$. In the classical context, we use the radial gauge more, whereas in the quantum description we use the symmetric, diagonal basis more. In order to avoid the introduction of further symbols, we use here and in all following formulas which refer to the Cartan subalgebra the convention i=1,2 and A=3,8. The transition from the radial gauge to the diagonal gauge is as follows. We can use the image of an SU(2) gauge transformation to turn the term $\hat{r}^s t^s(\beta)$ to $t^3(\beta)$. Explicitly

the action of this gauge transformation is given in terms of the matrix K

$$\mathrm{Ad}(g^{-1})t_j = K_{jk}t_k$$

with K given by

$$K_{ij} = \begin{pmatrix} 1-\frac{1}{2}(1-\cos\vartheta)(1+\cos 2\varphi) & -\frac{1}{2}(1-\cos\vartheta)\sin 2\varphi & \sin\vartheta\cos\varphi \\ -\frac{1}{2}(1-\cos\vartheta)\sin 2\varphi & 1-\frac{1}{2}(1-\cos\vartheta)(1-\cos 2\varphi) & \sin\vartheta\sin\varphi \\ -\sin\vartheta\cos\varphi & -\sin\vartheta\sin\varphi & \cos\vartheta \end{pmatrix}. \tag{6.1}$$

So the action of the gauge on the terms which appear in the formulas for BPS monopoles (3.18) is

$$\begin{aligned} \mathrm{Ad}(g^{-1})\hat{r}^s t^s &= t^3 \\ \mathrm{Ad}(g^{-1})(\delta_i^s - \hat{r}^s\hat{r}^i)t^s &= c_i^1 t_1 + c_i^2 t_2 = \\ &\begin{pmatrix} (1-\frac{1}{2}(1-\cos\vartheta)(1+\cos 2\varphi))t^1 - \frac{1}{2}(1-\cos\vartheta)\sin 2\varphi t^2 \\ -\frac{1}{2}(1-\cos\vartheta)\sin 2\varphi t^1 + (1-\frac{1}{2}(1-\cos\vartheta)(1-\cos 2\varphi))t^2 \\ -\cos\varphi\sin\vartheta t^1 - \sin\varphi\sin\vartheta t^2 \end{pmatrix} \end{aligned} \tag{6.2}$$

In this gauge the magnetic field is, in terms of the Abelian and non-Abelian component

$$B_i = B_{\mathrm{abel}}\hat{r}^i t^3 + B_{\mathrm{non}}(c_i^1 t_1 + c_i^2 t_2).$$

The Abelian and non-Abelian fields are the same as for SU(2), (5.21), (4.37). For large r only the Abelian part survives and goes as $-1/r^2$. This differs in sign from (3.15) and following.

The fields fields ϕ_1, ϕ_2 described in section 4.7, are the coefficients for the scalar field Φ in the diagonal basis

$$\Phi = \phi_1\mathcal{T}_1 + \phi_2\mathcal{T}_2 = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2-\phi_1 & 0 \\ 0 & 0 & -\phi_2 \end{pmatrix}. \tag{6.3}$$

The same holds for the dual field Φ_D.

Expression (3.18) for the scalar field Φ can be rewritten as

$$\begin{aligned} \Phi &= \left(h_A + (\phi - \vec{h}\cdot\vec{\beta})\beta_A^*\right)\frac{1}{\sqrt{3}}M_{Ai}^{-1}\mathcal{T}_i \\ &= \left(a_i + (\phi - \vec{h}\cdot\vec{\beta})\frac{n_{m,i}}{2}\right)\mathcal{T}_i, \end{aligned} \tag{6.4}$$

where we use

$$\beta_B^* = n_{m,j}M_{jB}\frac{\sqrt{3}}{2} \tag{6.5}$$

since the roots α_1 and α_3 are proportional to the columns of M.

As is shown explicitly later in section 6.4, all terms of $\mathcal{F}_{AB}$ with at least one index outside the Cartan subalgebra are zero. The form of the canonical momentum

is formally the same as for SU(2), given in (4.26); thus for large r it also lies in the Cartan subalgebra.
We define the electric and magnetic quantum numbers as

$$n_{e1}a_1 + n_{e2}a_2 = -2\int_{S_\infty} \mathrm{d}S^i \Pi_{iA}\phi^A \tag{6.6}$$

$$n_{m1}a_{D1} + n_{m2}a_{D2} = -\frac{1}{2\pi}\int_{S_\infty} \mathrm{d}S^i B_{iA}\phi_{D,A}. \tag{6.7}$$

The formula differs by a factor 2 from the formulas for SU(2) given in section 4.5, so as to comply with the definitions used in [21,24]. This also requires a fixed choice of simple roots as described in section 3.2. We use here the roots α_1 and α_3 (C-4) as simple roots.

The central charge is given as

$$Z = n_{m,1}a_{D,1} + n_{m,2}a_{D,2} + n_{e,1}a_1 + n_{e,2}a_2, \tag{6.8}$$

so, we choose as for SU(2) the constant α to be

$$\alpha = \frac{\pi}{2} - \arg Z. \tag{6.9}$$

This choice gives the correct mass of the monopole $m = \sqrt{2}|Z|$.
Similarly, as for SU(2), we can show that the phase of the generalized central charge $Z(r) = n_{m1}\phi_{D1} + n_{m2}\phi_{D2} + n_{e1}\phi_1 + n_{e2}\phi_2$ is constant. The procedure is almost the same as for SU(2), but we shall give here explicitly the intermediate steps. For the Abelian terms of the magnetic field and the conjugate momentum, we can find their asymptotic behavior from (6.6)

$$B_i \approx -\frac{\hat{r}^i}{r^2}(n_{m1}\mathcal{T}_1 + n_{m2}\mathcal{T}_2) \tag{6.10}$$

$$\Pi_i \approx -\frac{\hat{r}^i}{4\pi r^2}(n_{e1}\mathcal{T}_1 + n_{e2}\mathcal{T}_2) \tag{6.11}$$

It follows from the BPS equations (4.28) by simple manipulations that

$$B_i = -\sqrt{2}\mathrm{e}^{i\alpha}\nabla_i\Phi \tag{6.12}$$

$$\Pi_i = \frac{\sqrt{2}}{4\pi}\mathrm{e}^{i\alpha}\nabla_i\Phi_D, \tag{6.13}$$

these relations are analogous to (4.39), (4.40). From these, we find for the components in the Cartan subalgebra that

$$\begin{aligned}\sqrt{2}\nabla_i\mathrm{e}^{i\alpha}(n_{m1}\phi_{D1} + n_{m2}\phi_{D2} + n_{e1}\phi_1 + n_{e2}\phi_2) = \\ 4\pi n_{m1}\Pi_{i1} + 4\pi n_{m2}\Pi_{i2} - n_{e1}B_{i1} - n_{e2}B_{i2}.\end{aligned} \tag{6.14}$$

Using further the Gauss law $\nabla_i\Pi_{iA} = 0$ and the constraint $\nabla_i B_i^A = 0$, we get

$$\sqrt{2}\mathrm{e}^{i\alpha}\nabla_i\nabla_i Z(r) = 0.$$

The asymptotic form of the fields at infinity and the limit value at infinity imply that the central charge has a constant phase

$$\sqrt{2}\nabla_i\mathrm{e}^{i\alpha}Z(r) = 0.$$

6.2 Differential Equation for Moduli

We shall now derive a differential equation for the moduli u, v. The procedure is similar to the derivation of the SU(2) equation (5.7). We shall start with (6.14) and insert the explicit formulas for the magnetic field and the conjugate momentum. Explicitly the magnetic field component in Cartan subalgebra is

$$B^i_{\text{Cartan}} = \hat{r}^i B_{\text{abel}} t_3 = \hat{r}^i B_{\text{abel}} \vec{\beta^*} \vec{H} = \hat{r}^i B_{\text{abel}} \frac{1}{\sqrt{3}} \beta^*_A M^{-1}_{Ak} \mathcal{T}_k,$$

where M is the transition matrix (C-7). Thus we get

$$B^i_{\text{Cartan},k} = \hat{r}^i B_{\text{abel}} \frac{n_{\text{m},k}}{2}.$$

The conjugate momentum transforms in the same way as ϕ_D, (C-8), inserting the definition we find

$$\begin{aligned}\Pi^i_{\text{Cartan},k} = M_{kA}\Pi^i_A &= -\frac{\hat{r}^i}{4\pi\sqrt{3}} M_{kA} \mathcal{R}e \left((B_{\text{abel}} + iE_{\text{abel}}) \mathcal{F}_{AB} \vec{\beta^*_B} \right) \\ &= -\frac{\hat{r}^i}{4\pi} \mathcal{R}e \left((B_{\text{ab}} + iE_{\text{ab}}) \tau_{kj} \right) \frac{n_{m,j}}{2}.\end{aligned}$$

Here we used the fact that for nonzero terms both indices of $\mathcal{F}_{AB}$ must be in the Cartan subalgebra and relations (6.5) and (C-9) Inserting everything in (6.14) we find

$$\mathcal{R}e \left((B_{\text{abel}} + iE_{\text{abel}}) n_{m,j} \tau_{jk} n_{m,k} \right) + B_{\text{abel}} n_{e,k} n_{m,k} = 0,$$

which is the SU(3) analogue of (5.3). From this we find immediately, using the explicit form of the electric and magnetic fields, the analogue of (5.6)

$$b_r = \frac{1 - L^2}{r^2} \frac{n_{e,k} n_{m,k} + \mathcal{R}e\, n_{m,j} \tau_{jk} n_{m,k}}{\mathcal{I}m\, n_{m,j} \tau_{jk} n_{m,k}}.$$

Inserting this in the BPS equation for the Abelian component (3.11), we find

$$\sqrt{2} \mathrm{e}^{i\alpha} \phi_r = \frac{1 - L^2}{r^2} i \frac{n_{e,k} n_{m,k} + n_{m,j} \bar{\tau}_{jk} n_{m,k}}{\mathcal{I}m\, n_{m,j} \tau_{jk} n_{m,k}}.$$

Differentiating (6.4) with respect to r, we see that $\phi_{i,r} = n_{mi}\phi_r/2$; combining this with the chain rule, we find a first order differential equation for the moduli u, v

$$\begin{pmatrix} \frac{\mathrm{d}u}{\mathrm{d}r} \\ \frac{\mathrm{d}v}{\mathrm{d}r} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial \phi_1} & \frac{\partial u}{\partial \phi_2} \\ \frac{\partial v}{\partial \phi_1} & \frac{\partial v}{\partial \phi_2} \end{pmatrix} \begin{pmatrix} n_{m1} \\ n_{m2} \end{pmatrix} \frac{\phi_r}{2}. \tag{6.15}$$

Comparing this with (5.7), we see that this is analogous to the SU(2) case. As for SU(2) we can use the factor $\frac{1-L^2}{r^2}$ for a reparametrisation $r \to X(r)$, with $X(r)$ given in (4.44). Also, we can find equations for the phase and absolute value of the central charge. The central charge evolves according to

$$\frac{\mathrm{d}Z}{\mathrm{d}X} = \frac{i\mathrm{e}^{-i\alpha}}{\sqrt{2}} \frac{|n_{e,k} n_{m,k} + n_{m,j} \tau_{jk} n_{m,k}|^2}{\mathcal{I}m\, n_{m,j} \tau_{jk} n_{m,k}}. \tag{6.16}$$

Therefore the phase of the central charge is constant

$$\arg(Z) = \frac{\pi}{2} - \alpha \tag{6.17}$$

and the absolute value increases with increasing X

$$\frac{\mathrm{d}|Z|}{\mathrm{d}X} = \frac{1}{\sqrt{2}} \frac{|n_{e,k} n_{m,k} + n_{m,j} \tau_{jk} n_{m,k}|^2}{\mathcal{I}m\, n_{m,j} \tau_{jk} n_{m,k}}. \tag{6.18}$$

This is the same behavior as for SU(2), constant phase and an attractor at $Z = 0$. Since the reparametrisation is also the same, most of the general facts about the different types of behavior given in sections 5.2-5.4 are the same.

6.3 Numerical Solutions

As shown in appendix E, the period integrals ϕ_1, ϕ_2, $\phi_{D,1}$, $\phi_{D,2}$ and their derivatives are given in terms of the Appell function F_4 [24,40]. For values outside the radius of convergence, analytic continuations must be used, [40]. Unfortunately,not very many explicit formulas are available and the numerical calculation as a double sum is quite complicated because different representations must be used for different ranges of u,v. Rather than working out all representations for different cases, we chose to calculate the period integrals and their derivatives as contour integrals using their integral representation. We keep the notation from [24]. As said in section 4.6, all period integrals and their derivatives can be written as contour integrals of the holomorphic functions f, f_u, f_v respectively (4.49), (4.50):

$$\begin{aligned} f &= \frac{i}{2\pi} \frac{x(3x^2 - u)}{\sqrt{(x^3 - ux - v)^2 - \Lambda^6}} \\ f_u &= \frac{i}{2\pi} \frac{-x}{\sqrt{(x^3 - ux - v)^2 - \Lambda^6}} \\ f_v &= \frac{i}{2\pi} \frac{-1}{\sqrt{(x^3 - ux - v)^2 - \Lambda^6}}. \end{aligned} \tag{6.19}$$

The branch cuts of f and the cycles are chosen as in fig. 4.6. As in SU(2), the scale parameter Λ is set to one.

Roots

The equation of the elliptic curve (4.51) splits into a product of two cubic terms

$$y = (x^3 - ux - (v + \Lambda))(x^3 - ux - (v - \Lambda)).$$

The roots of a cubic polynomial can be found analytically using the standard Cardan formula. The roots of a monic cubic polynomial $x^3 + ax^2 + bx + c$, see e.g. [41]

$$
\begin{aligned}
z_1 &= -\frac{1}{3}\left(a + \sqrt[3]{\frac{R+\sqrt{R^2-4Q^3}}{2}} + \sqrt[3]{\frac{R-\sqrt{R^2-4Q^3}}{2}}\right)\\
z_2 &= -\frac{1}{3}\left(a + \mathrm{e}^{-i2/3\pi}\sqrt[3]{\frac{R+\sqrt{R^2-4Q^3}}{2}} + \mathrm{e}^{i2/3\pi}\sqrt[3]{\frac{R-\sqrt{R^2-4Q^3}}{2}}\right)\\
z_3 &= -\frac{1}{3}\left(a + \mathrm{e}^{i2/3\pi}\sqrt[3]{\frac{R+\sqrt{R^2-4Q^3}}{2}} + \mathrm{e}^{-i2/3\pi}\sqrt[3]{\frac{R-\sqrt{R^2-4Q^3}}{2}}\right),
\end{aligned}
\tag{6.20}
$$

where

$$R = 2a^3 - 9ab + 27c \qquad\qquad Q = a^2 - 3b.$$

In our case $R = 27(v \pm \Lambda)$, and $Q = 3u$; thus the term under the square root is proportional to the singular curve $27(v \pm \Lambda)^2 - 4u^3$. The roots of the elliptic curve are chosen so that in the limit $v \to 0$, $u \to \infty$ they give

$$
\begin{aligned}
e_1 &= -\sqrt{u} + \frac{1}{2}\frac{v}{u} - \frac{1}{2}\frac{\Lambda^3}{u} & e_2 &= -\sqrt{u} + \frac{1}{2}\frac{v}{u} + \frac{1}{2}\frac{\Lambda^3}{u}\\
e_3 &= -\frac{v}{u} - \frac{\Lambda^3}{u} & e_4 &= \frac{v}{u} - \frac{\Lambda^3}{u}\\
e_5 &= \sqrt{u} + \frac{1}{2}\frac{v}{u} - \frac{1}{2}\frac{\Lambda^3}{u} & e_6 &= \sqrt{u} + \frac{1}{2}\frac{v}{u} + \frac{1}{2}\frac{\Lambda^3}{u}
\end{aligned}
\tag{6.21}
$$

The roots on the left are roots of the polynomial with $v - \Lambda$, on the right those of the polynomial with $v + \Lambda$.

Cycles

As we move through moduli space, the roots e_i move as well and care is needed in the cycle algorithm. The simplest way to create a cycle is to take the points that we want to encircle and create a rectangular box, e.g. oriented parallel to the line which connects them. However, as the branchcuts move, we can reach points where this does not give the correct cycle anymore. More precisely, a function in this way, would not be continuous. This happens for both dual functions ϕ_{D1}, ϕ_{D2}, but the box-like path works fine for the integrals a_1, a_2.

We will illustrate this in the following example for ϕ_{D2}. We shall start at $u = 3$ and $v = 0$ and move along the real v axis with u fixed. The branchcuts and cycles (both correct and naive) are shown in fig. 6.1. For $v \in [0, 1]$, i.e. up to the closer singular, curve all six roots are real. As v increases, the branchcuts $e_1 - e_2$ and $e_3 - e_4$ move closer to each other, until they touch at $v = 1$ when the roots e_2 and e_3 become degenerate. For $v \in [1, 3]$ increasing, the branch cut between e_5 and e_6 becomes shorter while the the roots e_2, e_3 acquire an imaginary part of the same size but opposite sign and move away from each other (and the real axis). The roots e_1

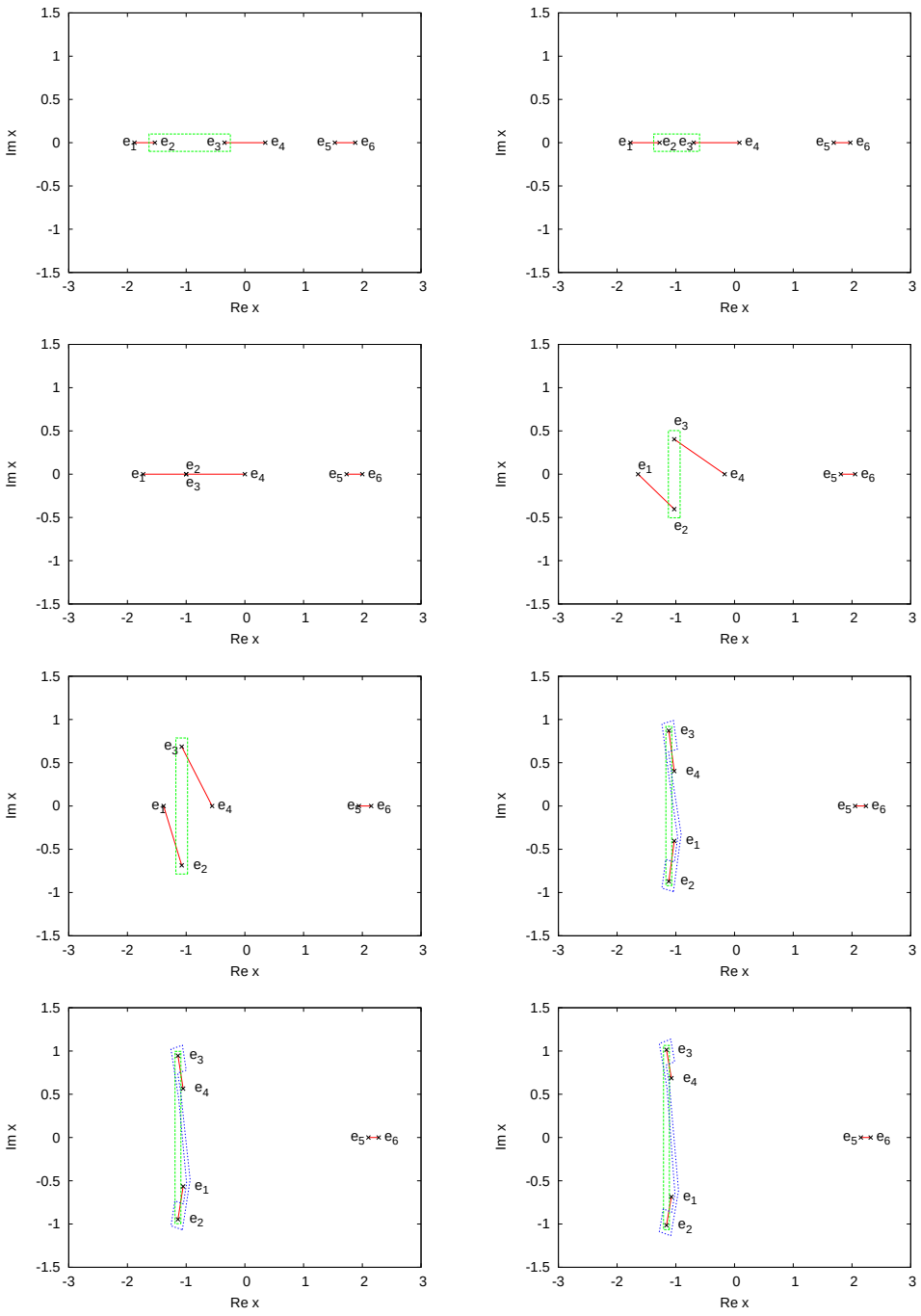

Figure 6.1: The roots e_i, $i = 1, \ldots, 6$ for $u = 3$ and $v \in [0, 5]$ in the x-plane. The branchcuts are shown in red. The integration path is green, if the correct cycle is different it is shown in blue.

and e_4 move closer to each other until they coincide at $v = 3$ on the second singular curve. For $v > 3$ the branchcuts $e_1 - e_2$ and $e_3 - e_4$ move symmetrically away from the real axis and their size decreases. Obviously for $v > 3$, the simple naive path

does not work anymore.
One possibility is to carefully define a different path, which would avoid the cuts and correctly encircle the roots e_2 and e_3 so that as we increase v the cycle is continuously deformed. However, we found this method quite cumbersome and difficult to implement in a fool-proof way. Instead we choose to use a combination of the naive path which is formed by a box around e_2,e_3 (except for points where roots become degenerate) and the integrals a_1, a_2. For the case, above the decomposition is shown in fig. 6.2. Thus we conclude that for the cycles (and integrals along them) the following relation holds

$$\beta_2 = \gamma - 2\alpha_2.$$

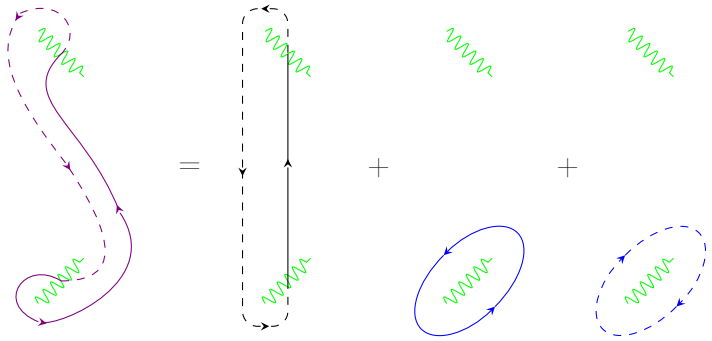

Figure 6.2: Decomposition of a more complicated cycle into simpler cycles and paths. In the example described in the text the cycle α_{D2} can be expressed in terms as $\alpha_{D2} = \gamma - 2\alpha_2$, where γ is the auxiliary cycle. The third branchcut is omitted for simplicity.

There remains another problem: if we are given a set of roots it is by far not obvious which path to choose so that the results will form a continuous function. If we stay in a reasonable region of the moduli space, we can hope to work things out correctly but one must always watch out for possible failures. Instead we choose to perform explicitly the "continuous deformation", i.e. we start at a known form of the path, given in fig. 4.6 and slowly move toward the desired point. At each step we check the relation between the path in the previous step and the current path. If, as in the case above, one of the roots "gets in the way" of our path we can compensate for that by adding/subtracting a suitable combination of a_1 (for the branchcut $e_5 - e_6$), (a_2 for the branchcut $e_1 - e_2$) and $a_1 - a_2$ (for the branchcut $e_3 - e_4$). The last path around $e_3 - e_4$ can be pulled across infinity and split into the two other paths, see fig. 6.3.
In some cases some of the roots lie on a single straight line and no box-like path which would encircle only the desired points can be found.

Again, we can create an auxiliary path, which encircles a different pair of roots, and write the desired cycle as a combination of this auxiliary path and α_i's, see fig. 6.4. The orientation comes up for a change of sign as well as the change between full line and dashed line, i.e. between the two sheets. Therefore we can change the

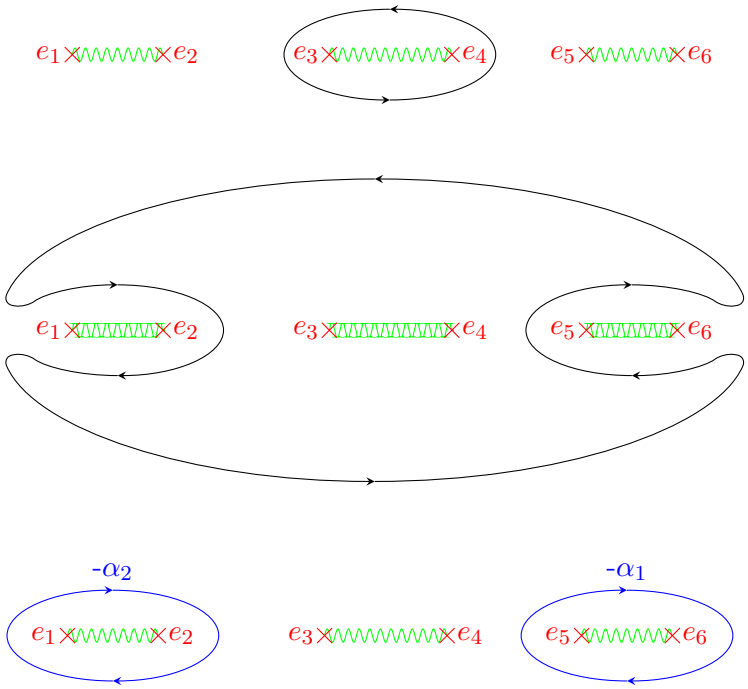

Figure 6.3: The cycle around the roots e_3, e_4 (top) can be deformed and split into three paths - around e_1, e_2, around e_5, e_6 and around infinity (middle), the loop around infinity can be pulled to a point leaving the cycles α_1, α_2.

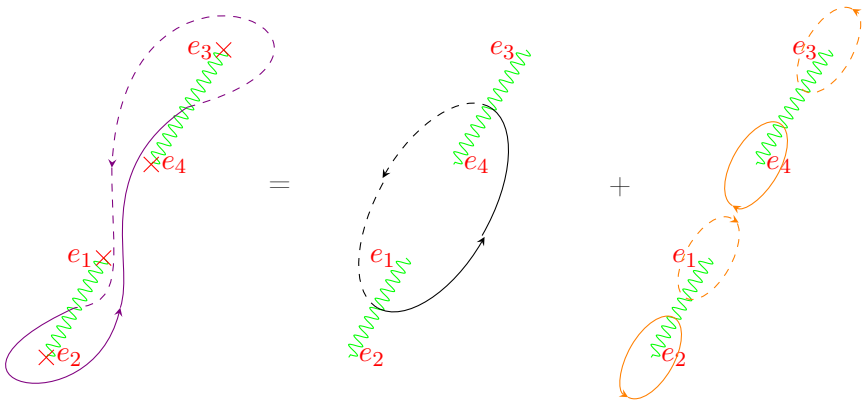

Figure 6.4: The cycle β_2 (violet) can be deformed and split into an auxiliary path γ encircling the pair (e_1, e_4) (black) and small auxiliary paths encircling each one of the four roots $e_i, i = 1, \ldots, 4$ (orange).

dashed paths into solid paths going in the opposite direction and combine them with the original ones to form the cycle $-\alpha_2$ and the path around e_3, e_4. The result is then the relation $a_{D2} = \gamma + \alpha_1 + 2\alpha_2$. The other cases can be worked out in the same way.

Integration

For the integration of the functions (6.19) along the chosen path, Simpson's adaptive rule is used, which is one of the simplest and most general methods. It is based on Simpson's rule for integration of a function $f(x)$ over an interval $[a, b]$

$$\int_a^b f(x)\mathrm{d}x = \frac{|b-a|}{3}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right).$$

The adaptive method splits the whole interval into two symmetric subintervals, which are each integrated according to Simpson's rule. This step is iterated until a specified precision is reached.

6.4 Curves of Marginal Stability

The derivation of the BPS equations assumed that Im $\mathcal{F}_{AB}$ is positive definite. This corresponds to several conditions, some of which define a separation of the moduli space in two parts, the border line is called curve of marginal stability. We will show in the following explicitly the form of the derivatives of the effective action $\mathcal{F}_{AB}$ and give all related formulas for easy reference. The derivatives were derived independently from [42], which uses a different and more general approach. The second derivative matrix $\mathcal{F}_{AB}$ is defined as

$$\mathcal{F}_{AB} = \frac{\partial^2 \mathcal{L}}{\partial\phi^A \partial\phi^B}.$$

Recall that we used the gauge invariant combinations of the field in the derivation of the SU(2) analogue (4.30). Whereas in SU(2) there is only one such object $\sum_A \phi^A\phi^A$, in SU(3) there are two such objects, the Weyl invariant Casimir variables defined in (4.45); in the standard basis, they can be written as

$$\begin{aligned} u &= \frac{1}{2}\mathrm{Tr}\Phi^2 = \frac{1}{4}\delta_{AB}\phi^A\phi^B \\ v &= \frac{1}{3}\mathrm{Tr}\Phi^3 = \frac{1}{6}d_{ABC}\phi^A\phi^B\phi^C, \end{aligned} \tag{6.22}$$

where d_{ABC} is the antisymmetric tensor given in (C-3). Then the most general form of $\mathcal{F}_{AB}$ is

$$\begin{aligned} \mathcal{F}_A &= \frac{1}{2}\partial_u\mathcal{F}\phi^A + \frac{1}{2}\partial_v\mathcal{F}d_{ABC}\phi^B\phi^C \\ \mathcal{F}_{AB} &= \partial_u\mathcal{F}\frac{1}{2}\delta_{AB} + \partial_v\mathcal{F}d_{ABC}\phi^C + +\frac{1}{4}\partial_u\partial_u\mathcal{F}\phi^A\phi^B + \\ &+ \frac{1}{4}\partial_u\partial_v\mathcal{F}(\phi^A d_{BCD} + \phi^B d_{ACD})\phi^C\phi^D + \\ &+ \frac{1}{4}\partial_v\partial_v\mathcal{F}d_{ACD}d_{BEF}\phi^C\phi^D\phi^E\phi^F. \end{aligned} \tag{6.23}$$

In the gauge used in section 6.1, where Φ lies only in the Cartan algebra, i.e. the only nonzero components of Φ are ϕ^3 and ϕ^8, we find that for indices outside the

Cartan subalgebra $\mathcal{F}_A$ is zero and $\mathcal{F}_{AB}$ is diagonal. The nonzero terms of $\mathcal{F}_A = \phi_{D,A}$ are

$$\begin{aligned}\phi_{D,3} &= \frac{1}{2}\partial_u\mathcal{F}\phi^3 + \frac{\sqrt{3}}{6}\partial_v\mathcal{F}\phi^3\phi^8 \\ \phi_{D,8} &= \frac{1}{2}\partial_u\mathcal{F}\phi^8 + \frac{\sqrt{3}}{12}\partial_v\mathcal{F}\left((\phi^3)^2 - (\phi^8)^2\right).\end{aligned} \tag{6.24}$$

This is analoguous to (4.29) for SU(2). The first terms are the direct analogues (differing only by a factor 1/2), the second terms do not have an analogue since there is no analogue of v and d_{ABC} for SU(2). The explicit form of $\phi_{D,i}$ in terms of ϕ_i can be found using (C-8)

$$\begin{aligned}\phi_{D,1} &= (2\phi_1 - \phi_2)(\partial_u\mathcal{F} + \partial_v\mathcal{F}\phi_2) \\ \phi_{D,2} &= (2\phi_2 - \phi_1)(\partial_u\mathcal{F} - \partial_v\mathcal{F}\phi_1),\end{aligned} \tag{6.25}$$

obviously it is more symmetric.
As for $\mathcal{F}_{AB}$, the terms on the diagonal are pairwise the same for the index pairs (1,2), (4,5) and (6,7) associated with each of the three roots α_i :

$$\begin{aligned}\mathcal{F}_{11} = \mathcal{F}_{22} &= \frac{1}{2}\partial_u\mathcal{F} + \frac{\sqrt{3}}{6}\partial_v\mathcal{F}\phi^8 \\ \mathcal{F}_{44} = \mathcal{F}_{55} &= \frac{1}{2}\partial_u\mathcal{F} + \partial_v\mathcal{F}\left(\frac{1}{4}\phi^3 - \frac{\sqrt{3}}{2}\phi^8\right) \\ \mathcal{F}_{66} = \mathcal{F}_{77} &= \frac{1}{2}\partial_u\mathcal{F} - \partial_v\mathcal{F}\left(\frac{1}{4}\phi^3 + \frac{\sqrt{3}}{2}\phi^8\right)\end{aligned} \tag{6.26}$$

These terms are analogous to the $\mathcal{F}'/\phi$ term in SU(2) (4.30). Since they contain only first derivatives, they can be rewritten in terms of the dual fields $\phi_{D,1}$, $\phi_{D,2}$ using (6.24):

$$\begin{aligned}\mathcal{F}_{11} = \mathcal{F}_{22} &= \frac{1}{2}\frac{\phi_{D,1}}{2\phi_1 - \phi_2} \\ \mathcal{F}_{44} = \mathcal{F}_{55} &= \frac{1}{2}\frac{\phi_{D,1} + \phi_{D,2}}{\phi_1 + \phi_2} \\ \mathcal{F}_{66} = \mathcal{F}_{77} &= \frac{1}{2}\frac{\phi_{D,2}}{2\phi_2 - \phi_1}\end{aligned} \tag{6.27}$$

The terms lying in the Cartan subalgebra are just a transformation of the period matrix $\tau_{ij} = \partial\phi_{D,i}/\partial\phi_j$ via (C-9).

When we use the embedded t'Hooft Polyakov monopole, we restrict ourselves, by choosing one specific root, to a certain subspace of the algebra. Therefore, depending on our choice of roots(i.e. magnetic numbers) we should use only the corresponding condition. The three roots and the corresponding subspaces are given in and above (C-4). In the diagonal gauge, the curves of marginal stabilities for the three different

roots are

$$\mathcal{C}_{10} : \mathcal{I}m \frac{\phi_{D,1}}{2\phi_1 - \phi_2} = 0 \tag{6.28}$$

$$\mathcal{C}_{11} : \mathcal{I}m \frac{\phi_{D,1} + \phi_{D,2}}{\phi_1 + \phi_2} = 0 \tag{6.29}$$

$$\mathcal{C}_{01} : \mathcal{I}m \frac{\phi_{D,2}}{2\phi_2 - \phi_1} = 0. \tag{6.30}$$

As can be seen from the explicit form of the fields (E-14), all three hyperplanes coincide at $v = 0$.

6.4.1 Large $|v|$ region

It was argued in [42] that for large v the topology of the curves of marginal stability should be $S^1 \times \mathbb{C}$, because the theory should behave as a copy of SU(2). We will describe this explicitly on our numerical solution for $\mathcal{C}_{01}$.
The region will be chosen as $\mathcal{R}e\, v > 1$ so that the Douglas-Argyres $\mathbb{Z}_3$ vacua $u = 0$, $v = \pm 1$ lie on the boundary. They can be found to lie on $\mathcal{C}_{01}$, as well as the $\mathbb{Z}_2$ vacua $u = \sqrt[3]{27/4}$ and $u = \sqrt[3]{27/4}\exp(\pm 2/3i\pi)$. The singular curves (4.52) where the discriminant becomes zero and particles become massless lie on the curve of marginal stability as well. Due to the expected topology, we choose a description which views the curve of marginal stability $\mathcal{C}_{01}$ on slices of constant v, focusing on the real part of v and its increase and treating the imaginary part of v as a fixed parameter. In this picture, the singular curves are for each slice only discrete points. For v real, larger than 1, $\mathcal{C}_{01}$ in the u-plane is approximately an ellipse with the major semiaxis on the positive real axis, see fig.6.5. In fact the major semiaxis is formed by the line which connects two points of the singular curve (4.52).

For a positive imaginary part of v we find that the position of the singular curve

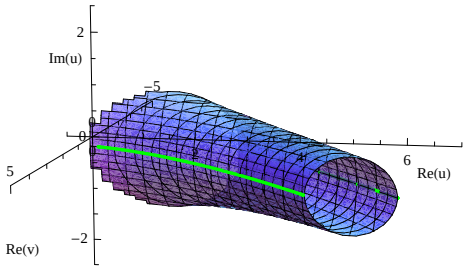

Figure 6.5: The curve of marginal stability $\mathcal{C}$ for magnetic numbers $(0, 1, 0, 0)$ restricted to v real, positive. The singular lines are shown in green.

points changes, see fig. 6.6. The curve of marginal stability $\mathcal{C}_{01}$ moves towards the upper half of the u-plane (i.e. the imaginary part increases). The orientation of the semiaxis changes as well - $\mathcal{C}_{01}$ tilts toward downwards.

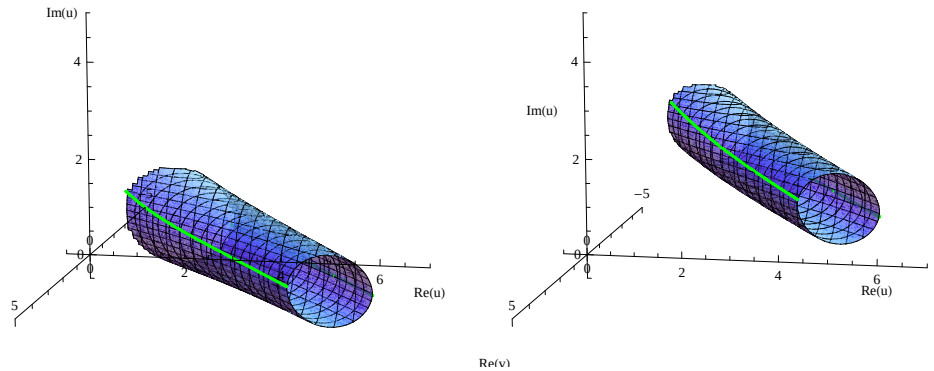

Figure 6.6: The curve of marginal stability $\mathcal{C}$ for magnetic numbers $(0,1,0,0)$ restricted to $\mathcal{I}m\,v = 1$ (left) and $\mathcal{I}m\,v = 3$ (right), for $\mathcal{R}e\,v$ positive. The singular lines are shown in green.

For $\mathcal{I}m\,v$ negative, the opposite happens, i.e. $\mathcal{C}_{01}$ moves into the lower half of the u-plane and turns upwards, see fig. 6.7.
Several curves of marginal stability are shown in fig. 6.8 for illustration.

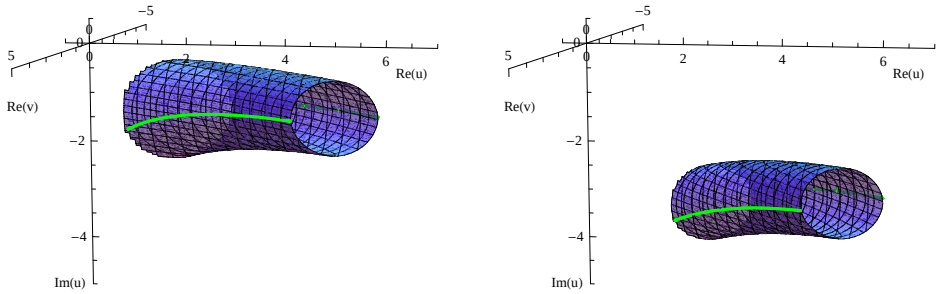

Figure 6.7: The curve of marginal stability $\mathcal{C}$ for magnetic numbers $(0,1,0,0)$ restricted to $\mathcal{I}m\,v = -1$ (left) and $\mathcal{I}m\,v = -3$ right) for $\mathcal{R}e\,v$ negative. The singular lines are shown in green.

A similar situation can be seen also for $\mathcal{R}e\,v$ negative. In this case, for real v or a positive imaginary part of v, the curve $\mathcal{C}_{01}$ is oriented along the line at an angle $2/3\pi$ with respect to the real u-axis. For a negative imaginary part, the curve is along a a line at an angle $-2/3\pi$. As can be seen in fig. 6.9, 6.10, 6.11, a positive imaginary part increases the phase and decreases the orientation whereas a negative imaginary part does the opposite. Several curves of marginal stability are shown together in fig. 6.12 for better comparison.

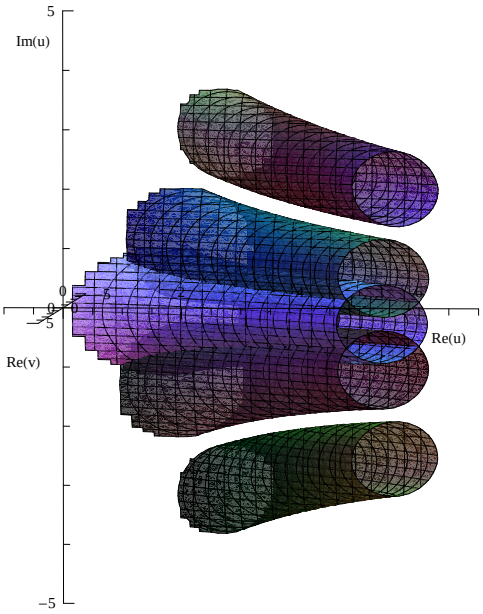

Figure 6.8: The curve of marginal stability $\mathcal{C}$ for magnetic numbers $(0,1,0,0)$, $\mathcal{R}e\, v$ positive and $\mathcal{I}m\, v = 3, 1, 0, -1, -3$ (from top to bottom).

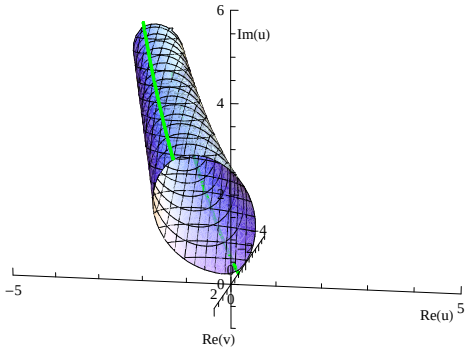

Figure 6.9: The curve of marginal stability $\mathcal{C}$ for magnetic numbers $(0,1,0,0)$ restricted to v real negative. The singular lines are shown in green.

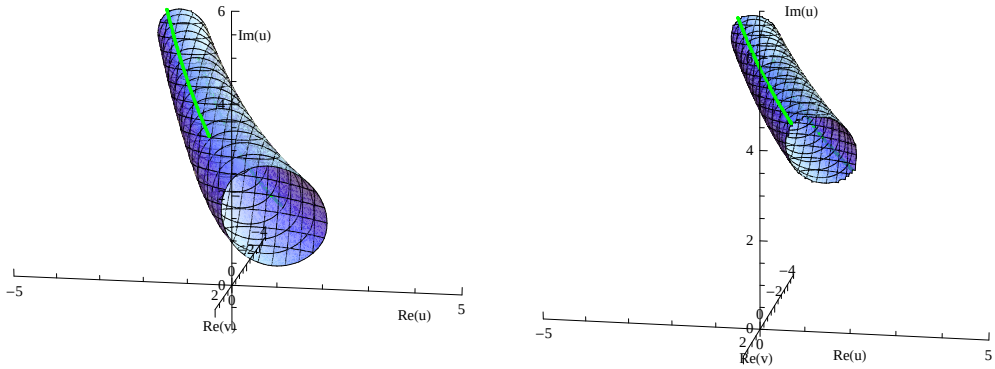

Figure 6.10: The curve of marginal stability $\mathcal{C}$ for magnetic numbers $(0, 1, 0, 0)$ restricted to $\mathcal{I}m\, v = 1$ (left) and $\mathcal{I}m\, v = 3$ (right), for $\mathcal{R}e\, v$ negative. The singular lines are shown in green.

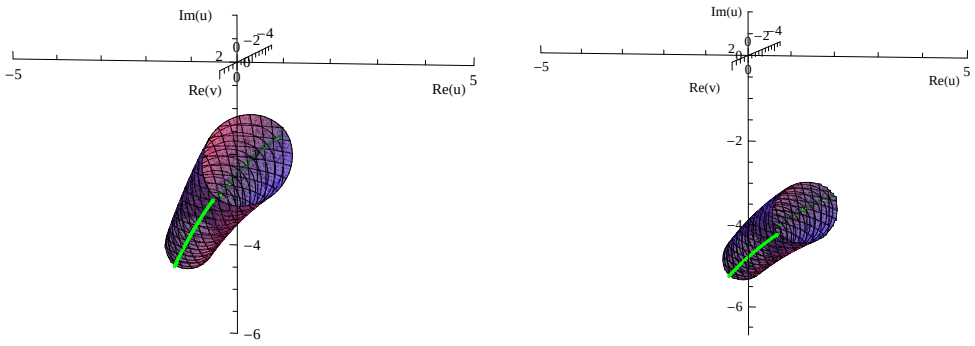

Figure 6.11: The curve of marginal stability $\mathcal{C}$ for magnetic numbers $(0, 1, 0, 0)$ restricted to $\mathcal{I}m\, v = -1$ (left) and $\mathcal{I}m\, v = -3$ (right), for $\mathcal{R}e\, v$ negative. The singular lines are shown in green.

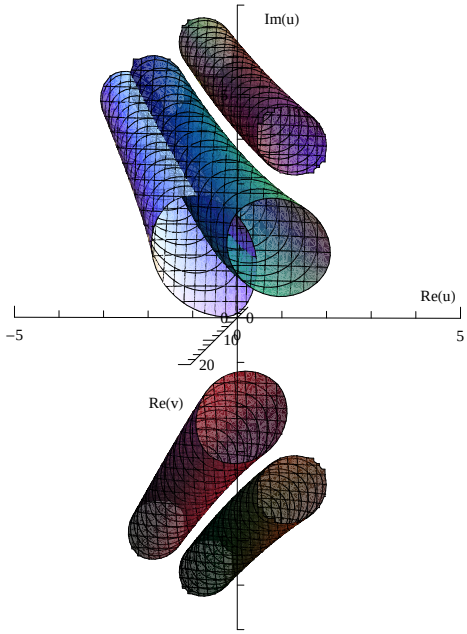

Figure 6.12: The curve of marginal stability $\mathcal{C}$ for magnetic numbers $(0, 1, 0, 0)$ and $\mathcal{I}m\, v = 3, 1, 0, -1, -3$ (from top to bottom), for $\mathcal{R}e\, v$ negative.

This behavior can be quantified by looking at the asymptotic behavior of the singular curve points. The singular curve in the u-plane for a fixed v, is formed by six points, with modulus $|u| = \sqrt[3]{\frac{27}{4}|v \pm 1|^2}$ and phase $\varphi = \frac{2}{3}\arg(v\pm 1), \frac{2}{3}\arg(v\pm 1)\pm\frac{2}{3}\pi$. Call the pair without the factor $\frac{2}{3}\pi$ $u_\pm$. For large v these points will behave as

$$u_\pm \approx \sqrt[3]{\frac{27}{4}} v^{2/3}\left(1 \pm \frac{2}{3}v^{-1}\right).$$

Thus the center of the approximate ellipse $S = \frac{1}{2}(u_+ + u_-)$ and the major semi-axis $a = \frac{1}{2}|u_+ - u_-|$ go as

$$S \approx \sqrt[3]{\frac{27}{4}} v^{2/3}\left(1 - \frac{1}{9}v^{-2}\right) \qquad r \approx \sqrt[3]{2}|v|^{-1/3}.$$

From this we see that the the curve $\mathcal{C}_{01}$ moves away from the origin as $|v|^{2/3}$ and under an angle of $\frac{2}{3}\arg(v)$, the extent of the curve decreases proportionally to $|v|^{-1/3}$. This behavior was verified numerically by fitting the curve $\mathcal{C}_{01}$ for a given v by an ellipse. We find that the predicted behavior for the center of the ellipse S, its major semi-axis a and its orientation are in good agreement (taking into account the fact that the curve is not a perfect ellipse). For large v, the orientation differs from the predicted one by 10^{-4}, the absolute value of the center by 10^{-3}, its phase by 10^{-6} and the major semi-axis by 10^{-3}. The minor semi-axis is found to decrease proportionally to the major semi-axis, in fact the ratio of the minor and major semi-axis approaches the value 0.86. This value is approximately the same as the one we would find for the $su(2)$ case shown in fig. 5.3.

6.5 Motion in moduli space

The moduli space of su(3) is $\mathbb{C}^2$, with "forbidden" regions - those inside the curves of marginal stability that cannot be described by our BPS solutions. The motion in moduli space is governed by eq. (6.15). The description of a general motion in four-dimensional space is difficult to view; for this reason we will give two examples in which the solutions are restricted only to a plane in $\mathbb{C}^2$.

6.5.1 Zero v plane

For $v = 0$, the functions ϕ_{D1} and ϕ_{D2}, resp. ϕ_1 and ϕ_2, coincide; from which it follows that the three possible curves of marginal stability $\mathcal{C}_{10}$, $\mathcal{C}_{01}$ and $\mathcal{C}_{11}$ also coincide. From the explicit form of the derivatives given in (E-16) it follows that their derivatives also satisfy the relation $\partial_u\phi_1 = \partial_u\phi_2$, $\partial_v\phi_1 = -\partial_v\phi_2$ and likewise for the duals. For the monopole $(1,1,0,0)$, we find from (6.15) that the derivative $\frac{\mathrm{d}v}{\mathrm{d}r}$ vanishes. Thus if we choose the value at infinity v_0 as zero, the moduli v will remain zero and the motion is only in the u-plane. The singular curves coincide, leaving only three, the $\mathbb{Z}_2$ vacua. As can be seen in fig. 6.13, the situation is very similar to su(2), which is described in detail in section 5.2-5.4. We find again solutions which

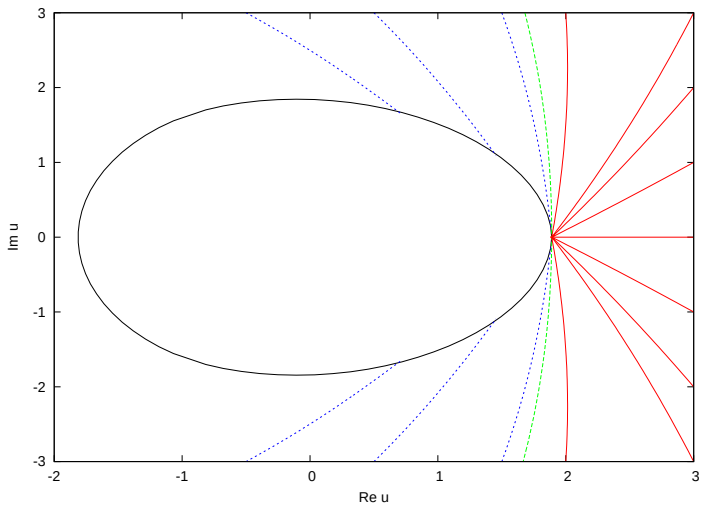

Figure 6.13: The curve of marginal stability $\mathcal{C}$ for magnetic numbers $(1,1)$ restricted to the $v = 0$ and solutions for different initial values. The green lines correspond to $\alpha = \pm\pi/2$, the red lines are Z-poles and the blue lines are X-poles.

hit the curve of marginal stability at one of the $\mathbb{Z}_2$ vacua (Z-poles), solutions which hit the curve of marginal stability somewhere else (X-poles), and the border between these two regions is given by solutions with a phase of the central charge $\alpha = \pm\pi/2$ (XZ-poles).

6.5.2 Real u,v plane

A nice example involving also the moduli v is the plane $(\mathcal{R}e\, u, \mathcal{R}e\, v)$. In the region below the singular curve, see fig. 6.14, the dual period ϕ_{D2} is purely imaginary whereas the periods ϕ_1, ϕ_2 are purely real. Thus for a monopole (0,1,0,0) the central charge has a zero phase and the solution remains in this plane. The curve of marginal stability $\mathcal{C}_{01}$ is shown with several solutions in fig. 6.14. All solutions are Z-poles. The region between the two parts of $\mathcal{C}_{01}$ cannot be accessed, since the BPS equations need not hold anymore. In the region closer to the real v axis, there are solutions, but they are no longer restricted to $(\mathcal{R}e\, u, \mathcal{R}e\, v)$ and they have been omitted for clarity.

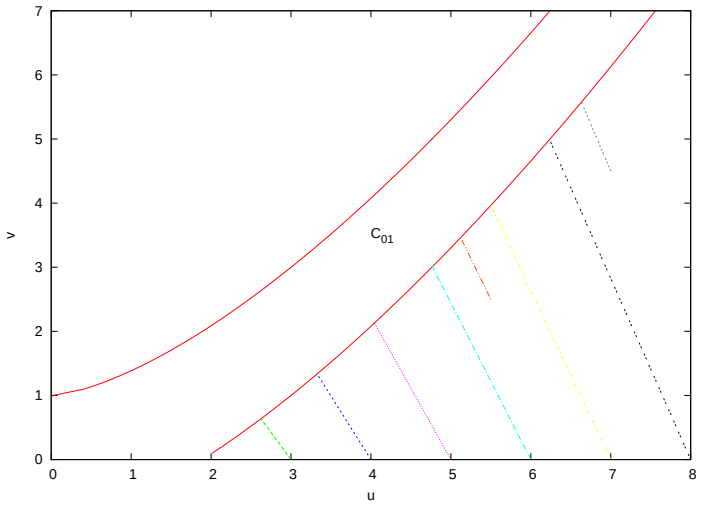

Figure 6.14: The curve of marginal stability $\mathcal{C}$ for magnetic numbers $(0, 1)$ restricted to u, v real and solutions for different initial values. All solutions shown are Z-poles.

Chapter 7

Conclusions

This work focused on studying the numerical behavior of period integrals and moduli in quantum corrected $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. Two gauge groups were studied: SU(2) and SU(3). These were chosen because they are the simplest non-Abelian groups with only one or two moduli, and because they had already been studied extensively in literature.

For SU(2), a first order differential equation was found which governs the spatial behavior of the modulus u. The whole analysis is valid only if the imaginary part of the matrix of second derivatives of the prepotential $\mathcal{I}m\,\mathcal{F}_{AB}$ is positive definite. This implies that we must stay outside the curve of marginal stability, which separates two regions with different spectra. Solutions were studied both numerically and analytically; the main focus was on the monopole solution with magnetic quantum number equal to 1 and zero electric quantum number. The differential equation is found to be an attractor equation, which means that for a given choice of electric and magnetic quantum numbers all solutions end at one single point in moduli space. Different types of behavior were found, depending on the choice of the free parameter δ and the initial condition $u_0 = u(\infty)$. The value of u_0 determines the overall shape of the solution in moduli space. It determines the phase of the central charge and whether the solution hits the curve of marginal stability (where our description ceases to be valid) or whether it goes all the way to the attractor. The value of δ determines the parametrisation of the solution with respect to the spatial parameters. For particular values the parametrisation can change its direction and the solution goes back on the same curve. All types of solutions, including their combinations, were described together with the electric and magnetic fields and the energy density. A special solution was identified which allows us to cut off the BPS monopole at a given radius and put a Higgs vacuum bubble inside.

For SU(3) there are two moduli, i.e. the moduli space is $\mathbb{C}^2$. This increases the complexity of all equations and makes any visualization almost impossible. Therefore we focused on those examples that could be fairly easily visualized in $\mathbb{R}^3$. The curves of marginal stability were found and studied numerically, showing how for large values of one of the two moduli the image of SU(2)'s curve of marginal stability appears. A system of two differential equations was found for the moduli. These equations have a similar structure to the SU(2) moduli equation: they are also attractor equations

and require the phase of the central charge to remain constant (which is a well-known result in SU(2)). The parametrisation involved is the same as in SU(2), so most of the analysis of the different solution types in SU(2) carries over to SU(3). For illustration, two examples were chosen such that the solution stays within a plane in the moduli space.

Appendix A

The hypergeometric and Appell functions

In this appendix we shall give briefly a few facts about the hypergeometric function and the Appell function, concentrating only on those relations that were used in our calculations. The hypergeometric function occurs for SU(2) solutions and also in certain limits of the Appell function, which is important for the solutions of SU(3). Detailed treatments can be found in [40, 43, 44].

Hypergeometric function

The hypergeometric series is given by

$$F(a,b,c,z) = \sum_{n=0} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \tag{A-1}$$

for $c \neq 0, -1, -2, \cdots$. Here, $(a)_n$ is the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

It is a solution of the Riemann equation

$$z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} f(z) + [c - (a+b+1)z]\frac{\mathrm{d}}{\mathrm{d}z} f(z) - abf(z) = 0. \tag{A-2}$$

The hypergeometric function is the analytic continuation of the hypergeometric series. The solution of the Riemann equation can be written in terms of the six so-called Kummer solutions, which can be found in the literature mentioned above. In our calculations we often need the behavior of the hypergeometric function at the singular points of the Riemann equation: $z = 0, 1$ and infinity. These are given by

$$F(a,b,c,0) = 1 \tag{A-3}$$

$$F(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{A-4}$$

The asymptotic behavior for $z \to \infty$ in the special case $b = a + m$ with m integer, is given by the following series

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(a+m)} \left[\frac{(-z)^{-a-m}}{\Gamma(c-a)} \sum_{n=0}^{\infty} \frac{(a)_{n+m}(1-c+a)_{n+m}}{n!(n+m)!} z^{-n} \right. \tag{A-5}$$

$$\left. [\log(-z) + h_n] + (-z)^{-a} \sum_{n=0}^{m-1} \frac{\Gamma(m-n)(a)_n}{\Gamma(c-a-n)n!} z^{-n} \right], \tag{A-6}$$

the coefficients are given by

$$h_n = \psi(1+m+n) + \psi(1+n) - \psi(a+m+n) - \psi(c-a-m-n),$$

where ψ is the logarithmic derivative of the Γ-function $\psi = \mathrm{d}\log\Gamma(z)/\mathrm{d}z$. Furthermore we used the relations

$$\begin{aligned}
\Gamma(z)\Gamma(1-z) &= \frac{\pi}{\sin \pi z} \\
\psi(1) &= -\gamma \\
\psi(z) &= \psi(1+z) - \frac{1}{z} \\
\psi\left(\frac{p}{q}\right) &= -\gamma - \log q - \frac{\pi}{2}\cot\frac{\pi p}{q} + {\sum_{n=1}^{n\le\frac{q}{2}}}' \cos\frac{2\pi n p}{q} \log\left(2 - 2\cos\frac{2\pi n}{q}\right) \\
& p, q \in \mathbb{N}, \quad p < q
\end{aligned}$$

where the prime in the last relation denotes that if q is even the last term in the sum should be multiplied by $\frac{1}{2}$.
The derivative of the hypergeometric function can be written in terms of a hypergeometric function with shifted parameters

$$\frac{\mathrm{d}}{\mathrm{d}z} F(a,b,c,z) = \frac{ab}{c} F(a+1, b+1, c+1, z). \tag{A-7}$$

There are many relations that relate hypergeometric functions with parameters shifted by ± 1, or functions with arguments $z, 1/z, 1-z, 1/(1-z), z/(1-z), 1-1/z$. We shall list only those that were used in the text

$$\begin{aligned}
0 = {} & cF(a,b-1,c,z) + (a-c)zF(a,b,c+1,z) + \\
& + (z-1)F(a,b,c,z)
\end{aligned} \tag{A-8}$$

$$F(a,b,c,z) = (1-z)^{-a} F\left(a, c-b, c, \frac{z}{z-1}\right) \tag{A-9}$$

$$\begin{aligned}
F(a,b,c,z) = {} & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} (-z)^{-a} F\left(a, a+1-c, a+1-b, \frac{1}{z}\right) + \\
& + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} F\left(b+1-c, b, b+1-a, \frac{1}{z}\right)
\end{aligned} \tag{A-10}$$

For some special values of the parameters, the hypergeometric function reduces to other functions. E.g. the relation between elliptic integrals and the hypergeometric function is useful for the simplification of the SU(2) periods

$$\begin{aligned} \mathrm{E}(k) &= \frac{\pi}{2} F(\frac{1}{2}, -\frac{1}{2}, 1, k^2) \\ \mathrm{K}(k) &= \frac{\pi}{2} F(\frac{1}{2}, \frac{1}{2}, 1, k^2) \end{aligned} \tag{A-11}$$

Appell functions

There are several generalizations of the hypergeometric function for two variables; in this work we need only the Appell function $F_4(a, b, c, c', x, y)$, which is given as a series by

$$F_4(a, b, c, c', x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_n (c')_m (1)_n (1)_m} x^n y^m. \tag{A-12}$$

This reduces to the hypergeometric function if one of the two arguments x, y becomes zero. Furthermore, the double sum can be reorganized

$$F_4(a, b, c, c', x, y) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} F(a+k, b+k, c', y) x^k \tag{A-13}$$

with an analogous relation with x, c and y, c' interchanged. From the form of F_4 it can be easily seen that its derivative is again an F_4 function, with different parameters

$$\frac{\mathrm{d}}{\mathrm{d}x} F_4(a, b, c, c', x, y) = \frac{ab}{c} F_4(a+1, b+1, c+1, c'; x, y),$$

and an analogous relation for y. The region of convergence is given by $|\sqrt{x}|+|\sqrt{y}| < 1$. There is a relation analogous to (A-10), which is used in the text

$$\begin{aligned} F(a, b, c, c', x, y) = {} & \frac{\Gamma(c')\Gamma(b-a)}{\Gamma(c'-a)\Gamma(b)} (-y)^{-a} F_4\left(a, a+1-c', c, a+1-b; \frac{x}{y}, \frac{1}{y}\right) + \\ & + \frac{\Gamma(c')\Gamma(a-b)}{\Gamma(a)\Gamma(c'-b)} (-y)^{-b} F_4\left(b+1-c', b, c, b+1-a; \frac{x}{y}, \frac{1}{y}\right), \end{aligned} \tag{A-14}$$

and a similar relation with x,y interchanged.
The Appell function $F_4(a, b, c, c'; x, y)$ fulfills the canonical equations (the parameters and arguments of F_4 have been suppressed)

$$\begin{aligned} & x(1-x)\partial_x^2 F_4 - y^2 \partial_y^2 F_4 - 2xy\partial_x\partial_y F_4 + [c - (1+a+b)x]\,\partial_x F_4 - \\ & \quad - (1+a+b)y\partial_y F_4 - abF_4 = 0 \\ & y(1-y)\partial_y^2 F_4 - x^2 \partial_x^2 F_4 - 2xy\partial_s\partial_y F_4 + \left[c' - (1+a+b)y\right]\partial_y F_4 - \\ & \quad - (1+a+b)x\partial_x F_4 - abF_4 = 0. \end{aligned} \tag{A-15}$$

The fundamental solutions can be chosen e.g.

$$
\begin{aligned}
z_1 &= F_4(a, b, c, c', s, t) \\
z_2 &= s^{1-c} F_4(a+1-c, b+1-c, 2-c, c', s, t) \\
z_3 &= t^{1-c'} F_4(a+1-c', b+1-c', c, 2-c', s, t) \\
z_4 &= s^{1-c} t^{1-c'} F_4(a+2-c-c', b+2-c-c', 2-c, 2-c', s, t);
\end{aligned} \tag{A-16}
$$

other forms can be used in order to achieve the desired region of convergence.

Appendix B

Expansions around $u = 1$

The scalar field ϕ and its dual ϕ_D are written in terms of elliptic integrals. The arguments of these go to 0, resp. 1, for $u = 1$. The elliptic integral $\mathrm{K}(q)$ diverges for $q \to 1$, so we must use expansions.

The general formulas for expansions of the elliptic integrals around $k = 0$ and $k = 1$, resp., are [43]

$$\text{for} \quad k \to 0$$

$$\mathrm{K}(k) = \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \ldots + \left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n} + \ldots\right\} \tag{B-1}$$

$$\mathrm{E}(k) = \frac{\pi}{2}\left\{1 - \frac{1}{2^2}k^2 - \ldots - \left[\frac{(2n-1)!!}{2^n n!}\right]^2 \frac{k^{2n}}{2n-1} - \ldots\right\} \tag{B-2}$$

$$\text{for} \quad k \to 1$$

$$\mathrm{K}(k) = \ln\frac{4}{k'} + \left(\frac{1}{2}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2}\right)(k')^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \right.$$
$$\left. - \frac{2}{3\cdot 4}\right)(k')^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4} - \frac{2}{5\cdot 6}\right)(k')^6 + \ldots \tag{B-3}$$

$$\mathrm{E}(k) = 1 + \frac{1}{2}\left(\ln\frac{4}{k'} - \frac{1}{1\cdot 2}\right)(k')^2 + \frac{1^2\cdot 3}{2^2\cdot 4}\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \frac{1}{3\cdot 4}\right)(k')^4 +$$
$$+\frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6}\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4} - \frac{1}{5\cdot 6}\right)(k')^6 + \ldots. \tag{B-4}$$

We shall expand u in the form $u = 1 + \varepsilon\, \mathrm{e}^{i\phi}$. For the field ϕ and its dual ϕ_D, we find the following expansions

$$\phi = \frac{4}{\pi}\left(1 - \frac{1}{8}\varepsilon\ln\varepsilon\mathrm{e}^{i\phi} + \left(\frac{5}{8}\ln 2 + \frac{1}{8}\right)\varepsilon\mathrm{e}^{i\phi} - \frac{1}{8}i\varphi\varepsilon\mathrm{e}^{i\phi} + \frac{1}{4}\varepsilon\mathrm{e}^{i\phi} + O(\varepsilon^2)\right) \tag{B-5}$$

$$\phi_D = i\frac{1}{2}\varepsilon\mathrm{e}^{i\phi}(1 - \frac{5}{32}\varepsilon\mathrm{e}^{i\phi}). \tag{B-6}$$

We can write the expansion of ϕ_D in the form $\phi_D \approx \frac{i}{2}\varepsilon\mathrm{e}^{i\phi}\mathrm{e}^{-\frac{5}{32}\varepsilon(\cos\varphi + i\sin\varphi)}$, so its phase is $\arg\phi_D = \frac{\pi}{2} + \varphi - \frac{5}{32}\varepsilon\sin\varphi$. This phase is constant along the solution of the

differential equation (5.7) and equal to $\frac{\pi}{2} - \alpha$. Thus we get a relation between ε and φ close to $u = 1$ for curves of constant Z-phase

$$\alpha + \varphi \approx \frac{5}{32}\varepsilon \sin\varphi. \tag{B-7}$$

From this we see that φ goes to $-\alpha$ as we get closer to $u = 1$.

We can find the differential equation for ε and solve it approximately to lowest order. Inserting $u = 1 + \varepsilon \mathrm{e}^{-i\alpha}$ in (5.10) we find the differential equation

$$\varepsilon_t = \frac{\pi}{2}\sqrt{2}\frac{\mathrm{e}^{-i\alpha}}{-\frac{1}{2}\ln\varepsilon}, \tag{B-8}$$

where we used $\mathrm{K}(q) \approx -\frac{1}{2}\ln\varepsilon$ and $i\bar{\tau}/\mathcal{I}m\,\tau \approx 1$. The solution of this equation is

$$\varepsilon \approx \frac{-\pi\sqrt{2}(t - t_0)}{\ln\left[-\pi\sqrt{2}(t - t_0)\right]}, \tag{B-9}$$

the constant t_0 is chosen so that $\varepsilon(t_0) = 0$, ie. for Z-poles and critical Z-poles $t_0 = t(r_c)$, for XZ-poles and bouncing XZ-poles $t_0 = 0$. Thus, close to $u = 1$ the solution goes as

$$u = 1 + \frac{-\pi\sqrt{2}(t - t_0)}{\ln[-\pi\sqrt{2}(t - t_0)]}\mathrm{e}^{-i\alpha}. \tag{B-10}$$

In terms of the parameter r this can be written (except for critical Z-poles)

$$u = 1 - \pi\sqrt{2}t_r(r_a)\frac{r - r_a}{\ln(r - r_a)}\mathrm{e}^{-i\alpha},$$

where r_a is the point at which $u = 1$. For critical Z-poles $t_r(r_a) = 0$, so we have to take a higher term and get $u = 1 - \pi\sqrt{2}\frac{1}{4}t_{rr}(r_a)(r - r_a)^2/\ln(r - r_a)$.

For the calculation of the electric fields we need the expansions of τ and $\frac{\phi_D}{\phi}$; these are

$$\tau = -\frac{i\pi}{\ln\varepsilon + O(\varepsilon^0)}\left(1 + \frac{1}{8}\varepsilon\mathrm{e}^{i\phi} + O(\varepsilon^2)\right) \tag{B-11}$$

$$\frac{\phi_D}{\phi} = \frac{i\pi}{8}\varepsilon\mathrm{e}^{i\phi} + O(\varepsilon^2), \tag{B-12}$$

Further, we need the following expansions

$$\frac{\mathcal{R}e\,\tau}{\mathcal{I}m\,\tau} = -\frac{-\frac{1}{8}\varepsilon\sin\alpha + O(\varepsilon^2)}{1 + \frac{1}{8}\varepsilon\cos\alpha + O(\varepsilon^2)} \tag{B-13}$$

$$\frac{\mathcal{R}e\,\frac{\phi_D}{\phi}}{\mathcal{I}m\,\frac{\phi_D}{\phi}} = \frac{i\pi}{8}\varepsilon\mathrm{e}^{-i\alpha} \tag{B-14}$$

$$\frac{\overline{\frac{\phi_D}{\phi}}}{\mathcal{I}m\,\frac{\phi_D}{\phi}} = -i(1 + i\tan\alpha) + O(\varepsilon). \tag{B-15}$$

Appendix C

SU(3) notations

The algebra $su(3)$ is well known, for a detailed treatment, see e.g. [45], [46]. The $su(3)$ algebra consists of traceless Hermitian 3×3 matrices. A convenient basis of these is constituted by the Gell-Mann matrices λ_a

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{C-1}$$

The generators are usually chosen to be $T_A = \frac{1}{2}\lambda_A$, so that the scalar product in $su(3)$ is $(T_A, T_B) = \mathrm{tr}(T_A T_B) = \frac{1}{2}\delta_{AB}$. We will refer to this as to the standard basis of SU(3). The structure constants are then by $[T_A, T_B] = i c_{ABC} T_C$

$$\begin{aligned} c_{345} = c_{246} = -c_{156} = c_{147} = c_{257} = -c_{367} &= \frac{1}{2} \\ c_{458} = c_{678} &= \frac{\sqrt{3}}{2} \\ c_{123} &= 1 \end{aligned} \tag{C-2}$$

The nonzero components of the totally symmetric $SU(3)$ tensor d_{ABC} are in the

standard basis (up to symmetry) $d_{ABC} = tr(\{T_A, T_B\}T_C)$,

$$\begin{aligned} d_{443} = d_{553} = d_{641} = d_{652} &= \frac{1}{4} \\ d_{663} = d_{773} = d_{742} = -d_{751} &= -\frac{1}{4} \\ d_{811} = d_{822} = d_{833} &= \frac{\sqrt{3}}{6} \\ d_{844} = d_{855} = d_{866} = d_{877} &= -\frac{\sqrt{3}}{12} \\ d_{888} &= -\frac{\sqrt{3}}{6}. \end{aligned} \tag{C-3}$$

.

Following the notation of [21] and [22], we will use a rescaled basis which brings the restriction of the Killing metric on the Cartan subalgebra to a unit matrix. The rescaling is $1/\sqrt{3}$ for the generators of the Cartan subalgebra which is chosen as usual in the directions T_3, T_8 and $1/\sqrt{6}$ for the root generators

$$\begin{aligned} H_1 &= \frac{1}{\sqrt{3}}T_3 & E_{\pm\alpha_1} &= \frac{1}{\sqrt{6}}(T_1 \pm iT_2) \\ H_2 &= \frac{1}{\sqrt{3}}T_8 & E_{\pm\alpha_2} &= \frac{1}{\sqrt{6}}(T_4 \pm iT_5) \\ & & E_{\pm\alpha_3} &= \frac{1}{\sqrt{6}}(T_6 \pm iT_7). \end{aligned}$$

The roots are then α_1, α_2 and α_3, all with the same norm $1/3$

$$\alpha_1 = \left(\frac{1}{\sqrt{3}}, 0\right) \quad \alpha_2 = \left(\frac{1}{2\sqrt{3}}, \frac{1}{2}\right) \quad \alpha_3 = \left(-\frac{1}{2\sqrt{3}}, \frac{1}{2}\right). \tag{C-4}$$

The simple roots can be chosen to be α_1 and α_3.

In terms of these rescaled Cartan and root generators, the nonzero components of the scalar product are

$$\begin{aligned} \mathrm{tr}(H_i H_j) &= \frac{1}{6}\delta_{ij} \\ \mathrm{tr}(E_{\alpha_j} E_{-\alpha_j}) &= \frac{1}{6}, \qquad j = 1, 2, 3. \end{aligned}$$

Another possible choice of basis for the Cartan subalgebra which treats the diagonal elements in a more symmetric way is

$$\mathcal{T}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{C-5}$$

$$\mathcal{T}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{C-6}$$

In the description of the quantum SU(3) monopole, we use both bases. The basis derived from the Gell-Man matrices is used for the parts which are closer to the classical description; the symmetric basis is used for matters related to the Seiberg-Witten description. The transition matrix from the standard basis to this symmetric basis is

$$M_{iA} = \begin{pmatrix} 2 & 0 \\ -1 & \sqrt{3} \end{pmatrix}. \tag{C-7}$$

The components of the field Φ, its dual Φ_D and the modulus τ transform in the usual way

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$$\begin{aligned} \phi_i &= \phi_A M^{-1}_{Ai} \\ \phi_{D,i} &= M_{iA}\phi_{D,A} \qquad \text{(C-8)} \\ \tau_{ij} &= M_{iA}\mathcal{F}_{AB}M^T_{Bj}. \qquad \text{(C-9)} \end{aligned}$$

Appendix D

General derivation of Picard Fuchs equations

Derivation of PF equations

In [37] a general, systematic method is presented, which can be used for all classical gauge groups with massless matter. The derivation is actually very simple, only algebraic operations and per-partes integrations are needed. But the cost for this is the large numbers of objects involved and the tedious calculations needed. We shall give here a very brief overview, focusing only on the main ideas, important formulas and (hopefully helpful) intermediate results. The elliptic curve for an SU(n) gauge group was given in (4.45), for the general case the second term reads as $x^k\Lambda^l$, the integers n, k, l are determined for the individual gauge groups and matter contents. The highest coefficient u_n can be normalized to 1 and the next highest coefficient can be set to zero $u_{n-1} = 0$. The scale Λ will be set to 1 for simplicity. The periods Π_j given in (4.48), (4.47) can be generalized to

$$\Omega_m^{(\mu)} = (-1)^{\mu+1}\Gamma(\mu+1)\oint_\gamma \frac{x^m}{W^{\mu+1}}\mathrm{d}x, \tag{D-1}$$

where $W = y^2$. The usual periods ω_j are obtained upon setting $\mu = -1/2$ and $m = 0, 1, \cdots, g-1$. The general idea of the derivation is as follows: both the periods $\Omega_m^{-1/2}$ and their derivatives with respect to the moduli $\partial\Omega_m^{-1/2}/\partial u_i$ can be expressed as linear combinations of the periods $\Omega_m^{1/2}$. This can be used to find a set of coupled first order differential equations for the periods $\Omega_m^{-1/2}$. This includes also the Seiberg-Witten potential λ_{SW}, since it is also a combination of these periods

$$\lambda_{SW} = \left[\frac{k}{2}p(x) - xp\prime(x)\right]\frac{\mathrm{d}x}{y}. \tag{D-2}$$

In the following, all total derivatives will be dropped. A period Ω_m^μ can be expressed in terms of the periods Ω_m^μ in two ways: either directly from the definition as

$$-(1+\mu)\Omega_m^m = (-1)^{\mu+2}\Gamma(\mu+2)\oint_\gamma \frac{x^m W}{W^{\mu+2}}$$

or by expressing x^{2n-1} from $\partial W/\partial x$, inserting in $\Omega^{\mu+1}_{m+2n-1}$ and using per partes to change the integral of $\partial W/\partial x$ to Ω^{μ}_{m}. Combining both we can find two crucial recursion relations for the periods

$$\begin{aligned}\Omega^{\mu}_{m} =& \frac{1}{m+1-2n(1+\mu)}\Big[(k-2n)\Omega^{\mu+1}_{m+k}+ \\ &+\sum_{j=0}^{n-1}\sum_{l=0}^{n-1}(2n-j-l)u_j u_l \Omega^{\mu+1}_{m+j+l}+2\sum_{j=0}^{n-1}(n-j)u_j\Omega^{\mu+1}_{m+n+j}\Big] \qquad \text{(D-3)}\\ \Omega^{\mu+1}_{m} =& \frac{1}{m+1-2n(\mu+2)}\Big[(m-2n+1-k(1+\mu))\Omega^{\mu+1}_{m+k-2n}+ \\ &+2\sum_{j=0}^{n-1}((1+\mu)(n+j)-(m-2n+1)u_j\Omega^{\mu+1}_{m-n+j}+ \\ &+\sum_{j=0}^{n-1}\sum_{l=0}^{n-1}((j+l)(1+\mu)-(m-2n+1))u_j u_l\Omega^{\mu+1}_{m+j+l-2n}\Big] \qquad \text{(D-4)}\end{aligned}$$

These recursion relations allow us to express the periods $\Omega^{(-1/2)}_{m}$ with m in the so-called basic range $R=(0,1,\ldots,g-1,g+1,\ldots,2g)$ as linear combinations $\Omega^{(1/2)}_{m}$, $m\in R$. We will discuss later the special cases $m=n-1$ and $m=3n-1$, where the recursion relations are not valid. Define the matrix of this linear transformation as M

$$\Omega^{-1/2}_{m}=M_{mm\prime}\Omega^{1/2}_{m\prime}, \qquad m,m\prime\in R. \qquad \text{(D-5)}$$

The linear expansion of the derivatives of the periods $\partial\Omega^{\mu}_{m}/\partial u_i$ in terms of the periods $\Omega^{\mu+1}_{m}$ can be found immediately from the definition

$$\frac{\partial\Omega^{\mu}_{m}}{\partial u_i}=2\sum_{j=0}^{n}u_j\Omega^{\mu+1}_{m+j+i}. \qquad \text{(D-6)}$$

The matrix of this linear transformation is $D(u_i)$ and we find the differential equation for the periods

$$\frac{\partial\Omega^{-1/2}_{m}}{\partial u_i}=D(u_i)_{mm\prime}\Omega^{1/2}_{m\prime}=\left(D(u_i)M^{-1}\right)_{mm\prime}\Omega^{-1/2}_{m\prime}. \qquad \text{(D-7)}$$

It can be proven that M is invertible everywhere except for the singular curve. Since we are interested only in the differential equation for the Seiberg-Witten potential, we can change our basis so that λ_{SW} is one of the basis vectors. This can be always done, since from (D-2) we see that as long as $k<2n$ (which is always fulfilled), λ_{SW} always contains a nonzero term proportional to ω_n. Denote the new basis formed by ω_m, $m\in R, m\neq n$ as π and the transition matrix from the old ω_m basis to the π basis as K. Then the differential equations for the new basis read

$$\frac{\partial\Pi^{-1/2}_{m}}{\partial u_i}=U_{i,mm\prime}\Pi^{-1/2}_{m\prime}, \qquad \text{(D-8)}$$

$$U_i=KD(u_i)M^{-1}K^{-1}+\frac{\partial K}{\partial u_i}K^{-}1. \qquad \text{(D-9)}$$

The extraction of a set of differential equations fulfilled by the SW potential is non-trivial. However if the SW potential is a potential for some differentials, i.e.

$$\frac{\partial \lambda_{SW}}{\partial u_i} = \Pi_i,$$

the situation can simplify. This is the case for $SU(n)$, and partly for $SO(n)$ and $Sp(n)$. In the case of the $SU(n)$, groups the SW potential acts as a potential for all differentials of the first kind, i.e. ω_m, $m < n-1$. The square $2g \times 2g$ matrix U_i can be split into four $g \times g$ matrices A_i, B_i, C_i, D_i

$$U_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}. \tag{D-10}$$

Solving for $\lambda_{SW} = \Pi_n$ from the first n equations, we find

$$\frac{\partial}{\partial u_i}\Pi_n = \sum_{k,l=1}^{n-1} \left[B_i^{-1}\right]_{1,k} \left(\frac{\partial \Pi_k}{\partial u_i} - A_{i,k,l}\Pi_l\right)$$

where we assumed that B_i is invertible. Using the fact that the periods Π_k, $k = 1, \ldots, n-1$ are derivatives of the SW potential we find the final equations

$$\sum_{k,l=1}^{n-1} \left[B_i^{-1}\right]_{1,k} \left(\frac{\partial^2 \lambda_{SW}}{\partial u_i \partial u_k} - A_{i,k,l}\frac{\partial \lambda_{SW}}{\partial u_l}\right) - \lambda_{SW} = 0. \tag{D-11}$$

At last, we will return to the problem cases when the recursion formulae (D-4) are not valid. The case $n-1$ does not pose a problem, since the period $\Omega_{n-1}^{(-1/2)}$ is not needed anyway. On the other hand, $\Omega_{3n-1}^{1/2}$ is needed to express both $\Omega_{2n-2}^{(-1/2)}$ and its derivatives. In order to find an alternative expression for $\Omega_{2n-2}^{(-1/2)}$, divide $x^{2n-2}W$ by $\partial W/\partial x$, denoting the quotient as $q(x)$ and the remainder as $r(x)$

$$x^{2n-2}W = q(x)\frac{\partial W}{\partial x} + r(x).$$

Integrating this, we find the relation

$$-(1+\mu)\Omega_{2n-2}^{(\mu)} = -\sum_{j=0}^{2n-1} j q_j \Omega_{j-1}^{(\mu)} + \sum_{l=0}^{2n-2} r_l \Omega_l^{(1+\mu)},$$

which can be used to find the following expression for $\Omega_{2n-2}^{(-1/2)}$

$$\Omega_{2n-2}^{(-1/2)} = \frac{2n}{n-1}\sum_{j=0}^{2n-2} r_l \Omega_l^{(1/2)} - l q_l \Omega_{l-1}^{(-1/2)}. \tag{D-12}$$

All occurrences of the period $\Omega_{n-1}^{(-1/2)}$, which is not in the standard range, should be moved to the left side, and the subspace formed this way should be used everywhere

instead of $\Omega_{2n-2}^{(-1/2)}$. A similar trick can be done for the derivatives, using $x^{2n-2}\partial W/\partial u_i$ instead of W. The result is

$$\frac{\partial\Omega_{2n-2}^{(-1/2)}}{\partial u_i} = -\sum_{j=0}^{n-1+i} jq_j^{(i)}\Omega_{j-1}^{(-1/2)} + \sum_{j=0}^{2n-1} r_l^{(i)}\Omega_l^{(1/2)}, \tag{D-13}$$

where $q_l^{(i)}$ are the coefficients of the quotient polynomial and $r^{(i)}$ the coefficients of the remainder. The coefficients of the quotient and remainder are derivatives of the corresponding coefficients of $q(x)$ and $r(x)$; therefore all relations refer to the same subspace. Also, $\Omega_{n-1}^{(1/2)}$ appeared as a side product in our relations, and thus we need some other relation to get rid of it. This can be achieved by doing the same trick for $x^{n-1}W$

$$-(1+\mu)\Omega_{n-1}^{(\mu)} = -\sum_{j=0}^{n} jq_j\Omega_{j-1}^{(\mu)} + \sum_{l=0}^{2n-2} r_l\Omega_l^{(1+\mu)}.$$

Furthermore, the leading coefficient of $q(x)$ must be $1/2n$; thus the problematic $\Omega_{n-1}^{(-1/2)}$ drops out and leaves us, after using (D-4), with a relation among the $\Omega_m^{(1/2)}$.

PF equations for SU(2)

For SU(2), the curve is given by (4.15); thus $n = 2, k = 0$, there is only one modulus $u_0 \equiv u$ and the standard range is $R = 0, 2$. The Seiberg-Witten potential is then $\lambda_{SW} = -2\Omega_2^{-1/2}$. The matrices M and $D(u)$ are readily found

$$M = 4\begin{pmatrix} 1-u^2 & -u \\ -u(1-u^2) & -(1-u^2) \end{pmatrix} \qquad D(u) = 2\begin{pmatrix} u & 1 \\ u^2-1 & u \end{pmatrix}. \tag{D-14}$$

From this we find

$$\begin{pmatrix} \partial_u\Omega_0^{(-1/2)} \\ \partial_u\Omega_2^{(-1/2)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & \frac{1}{u^2-1} \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \Omega_0^{(-1/2)} \\ \Omega_2^{(-1/2)} \end{pmatrix}, \tag{D-15}$$

which gives us the Picard-Fuchs equation

$$\frac{\mathrm{d}^2\lambda_{SW}}{\mathrm{d}u^2} = \frac{1}{4(1-u^2)}\lambda_{SW}. \tag{D-16}$$

PF equations for SU(3)

The elliptic curve for SU(3) is given by (4.51), the moduli are $u_0 = -u$, $u_1 = -v$, and the other parameters are $n = 3$, $k = 0$, $R = 0, 1, 3, 4$ and $\lambda_{SW} = u\Omega_1^{(-1/2)} - 3\Omega_3^{(-1/2)}$. Since the equations (D-4) and (D-6) for $\Omega_4^{(-1/2)}$ and its u-derivative involve $\Omega_8^{(1/2)}$, we must use the alternative relations (D-12), (D-13) and (D). The quotients, resp.,

polynomials are

$$\begin{aligned} q &= \frac{1}{6}x^5 - \frac{1}{9}ux^3 - \frac{1}{6}vx^2 - \\ &\quad - \frac{1}{27}u^2x - \frac{1}{18}uv \\ q^{(u)} &= -\frac{1}{3}x^3 - \frac{1}{9}ux \\ \tilde{q} &= \frac{1}{6}x^3 - \frac{1}{9}ux - \frac{1}{6}v \end{aligned} \qquad \begin{aligned} r &= -x^4(1 + \frac{2}{27}u^3) - \frac{1}{9}u^2vx^3 + \frac{2}{27}u^4x^2 + \\ &\quad - \frac{5}{27}u^3tx + \frac{1}{9}v^2u^2 \\ r^{(u)} &= -\frac{2}{9}u^2x^4 + \frac{2}{9}u^3x^2 - \frac{2}{9}u^2tx \\ \tilde{r} &= -\frac{2}{9}u^2x^4 - \frac{1}{3}uvx^3 - x^2(1 - \frac{2}{9}u^3) - \\ &\quad + \frac{5}{9}u^2vx + \frac{1}{3}uv^2 \end{aligned}$$

So the linear combination of $\Omega^{(-1/2)}$ which should be used instead of $\Omega_4^{(1/2)}$ is $\Omega_4^{(-1/2)} - u\Omega_2^{(-1/2)}$ and the auxiliary relation among the $\Omega^{(1/2)}$ is

$$\Omega_2^{(1/2)} = \frac{u}{3}\Omega_0^{(1/2)}.$$

The new basis is

$$(\Pi_1, \Pi_2, \Pi_3, \Pi_4) = (\Omega_0^{(-1/2)}, \Omega_1^{(-1/2)}, \Omega_3^{(-1/2)}, \Omega_4^{(-1/2)} - u\Omega_2^{(-1/2)})$$

and the matrices M, K, $D(v)$ and $D(u)$ are

$$M = \begin{pmatrix} -\frac{2}{3}u^3 - 3v^2 + 3 & 5uv & 3v & 2u \\ \frac{10}{3}u^2v & -\frac{4}{3}u^3 - 6v^2 + 6 & -\frac{4}{3}u^2 & 6v \\ -4u^3v + 3v - 3v^3 & 8u - 11uv^2 - \frac{4}{3}u^4 + 6v & \frac{4}{3}u^3 + 3v^2 - 6 & 8uv \\ -\frac{10}{3}u^2v^2 + \frac{1}{3}u^2 & -\frac{4}{3}u^3v - 6v^3 - 6v & \frac{4}{3}u^2v & -3 + 6v^2 \end{pmatrix}$$

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & u & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$D(t) = \begin{pmatrix} 2v & 2u & -2 & 0 \\ \frac{2}{3}u^2 & 2v & 0 & -2 \\ -v^2 + 1 & -uv & v & 0 \\ -\frac{8}{3}u^2v & -\frac{4}{3}u^3 + 4 - 4v^2 & \frac{4}{3}u^2 & 4v \end{pmatrix}$$

$$D(u) = \begin{pmatrix} -\frac{2}{3}u^2 & 2v & 0 & 2 \\ \frac{2}{3}uv & \frac{2}{3}u^2 & -\frac{2}{3}u & 0 \\ -2u^2v & -\frac{2}{3}u^3 + 4 - 4v^2 & \frac{2}{3}u^2 & 4v \\ -\frac{1}{3}uv^2 + \frac{1}{3}u + 1 - v^2 & -\frac{1}{3}u^2v - uv & \frac{1}{3}uv + v & 0 \end{pmatrix}.$$

The determinant of the matrix M is

$$\frac{4}{9}(27(1 - v)^2 - 4u^3)(27(1 + v)^2 - 4u^3)$$

so it can be inverted everywhere except for the singular curve. The SW differential is a potential for Π_1, Π_2, which is also reflected in the fact that the third lines of $U(v)$, resp., $U(u)$ are zeros except for the first, resp., second entries, which are ones. Performing the decoupling procedure described above, we find two differential equations $\mathcal{L}_i \lambda_{SW} = 0$

$$\mathcal{L}_1 = (27 - 4u^3 - 27v^2)\frac{\partial^2}{\partial u^2} - 12u^2 v \frac{\partial^2}{\partial u \partial v} - 3uv\frac{\partial}{\partial v} - u \tag{D-17}$$

$$\mathcal{L}_2 = (27 - 4u^3 + 27t^2)\frac{\partial^2}{\partial u^2} - 36uv \frac{\partial^2}{\partial u \partial v} - 9v\frac{\partial}{\partial v} - 3. \tag{D-18}$$

Appendix E

Solution of the Picard-Fuchs equations for SU(3)

The Picard Fuchs equations for SU(3) can be rewritten so that they form canonical equations for the Appell function F_4 [24]. The solution of the canonical equations can be found in [40].
Asymptotic behaviour can be found for the period integrals (4.50). We can then choose a suitable combination of fundamental solutions with the given asymptotics. The calculations are described in more detail here, the final results can be found in [24].
The first derivatives of the periods are given for further reference.

Differential equations

The Picard-Fuchs equations for the period integrals (4.54) can be rewritten in the canonical form (A-15) as follows. It is useful to change coordinates from (u, v) to coordinates $s = \frac{v^2}{\Lambda^6}$ and $t = \frac{4u^3}{27\Lambda^6}$. The operators $\mathcal{L}_1$ and $\mathcal{L}_2$ transform to

$$\begin{aligned}\mathcal{L}_1 &= 36u\left[t(1-t-s)\partial_t^2 - 2st\partial_s\partial_t + \frac{2}{3}(1-s-t)\partial_t - \frac{1}{6}s\partial_s - \frac{1}{36}\right]\\ \mathcal{L}_2 &= 108\left[s(1-s-t)\partial_s^2 - 2st\partial_s\partial_t + \frac{1}{2}(1-s-t)\partial_s - \frac{1}{6}s\partial_s - \frac{1}{36}\right].\end{aligned} \tag{E-1}$$

The condition (4.55) can also be transformed and gives

$$4\partial_t + 6t\partial^2 + t = 3\partial_s + 6s\partial_s^2. \tag{E-2}$$

We can use this condition to rewrite the operators $\mathcal{L}_1$, resp., $\mathcal{L}_2$ and get rid of the terms $st\partial_t^2, s\partial_t$ resp. $st\partial_s^2, t\partial_s$. The resulting form is

$$\begin{aligned}\mathcal{L}_1 &= t(1-t)\partial_t^2 - s^2\partial_s^2 - 2st\partial_s\partial_t + \left(\frac{2}{3} - \frac{2}{3}t\right)\partial_t - \frac{2}{3}s\partial_s - \frac{1}{36}\\ \mathcal{L}_2 &= s(1-s)\partial_s^2 - t^2\partial_t^2 - 2st\partial_s\partial_t + \left(\frac{1}{2} - \frac{2}{3}s\right)\partial_s - \frac{2}{3}t\partial_t - \frac{1}{36}.\end{aligned} \tag{E-3}$$

So the Picard-Fuchs equations are equivalent to the canonical equations for the Appell function $F_4(-\frac{1}{6}, -\frac{1}{6}, \frac{1}{2}, \frac{2}{3}, s, t)$. The fundamental solutions of these equations are given in (A-16), we can use these to match the asymptotic formulas found in [24].

Unfortunately, we do not get all the asymptotic behavior that we need. If we look at the $v \to 0$, $u \to \infty$ limit, which corresponds to $s \to 0$, $t \to 0$, we find, using $F_4(a, b, c, c', 0, t) = F(a, b, c', t)$ and the expansion formula for the (ordinary) Gaussian hypergeometric function, given in (A-5), for the solutions z_1 and z_3 the same behaviour, namely

$$z_1 \approx \text{const}\sqrt{u}(\log u + \text{const})$$

but no $\sqrt{u}$ behaviour that we need for the a_1, a_2 integrals, as we will see later. We can use a different solution, where F_4 has a different radius of convergence, namely $t^{-a}F_4(a, a+1-c', c, a+1-b, \frac{s}{t}, \frac{1}{t})$, (compare this with (A-14). Substituting this instead of z_3, z_4 we start with the following fundamental solutions of our system of equations

$$\begin{aligned}
z_1 &= F_4\left(-\frac{1}{6}, -\frac{1}{6}, \frac{1}{2}, \frac{2}{3}, s, t\right) \\
z_2 &= s^{\frac{1}{2}}F_4\left(\frac{1}{3}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, s, t\right) \\
z_3 &= t^{\frac{1}{6}}F_4\left(-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, 1, \frac{s}{t}, \frac{1}{t}\right) \\
z_4 &= s^{\frac{1}{2}}t^{-\frac{1}{3}}F_4\left(\frac{1}{3}, \frac{2}{3}, \frac{3}{2}, 1, \frac{s}{t}.\frac{1}{t}\right)
\end{aligned} \tag{E-4}$$

The expansions of these for large u (i.e. small t) and small v (and s) can be found using the expansion of hypergeometric function and the relations given in section 4.3

$$\begin{aligned}
z_1 &\approx \frac{\Gamma(\frac{2}{3})}{\Gamma(-\frac{1}{6})\Gamma(\frac{5}{6})}(-t)^{\frac{1}{6}}\left[\ln t + \pi(i - \sqrt{3}) + 3\ln 3 + 4\ln 2 - 6\right] \\
z_2 &\approx (-1)^{-\frac{1}{3}}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})^2}s^{\frac{1}{2}}t^{-\frac{1}{3}}\left[\ln t + \pi(i + \frac{1}{\sqrt{3}}) + 3\ln 3\right] \\
z_3 &\approx t^{\frac{1}{6}} \\
z_4 &\approx s^{\frac{1}{2}}t^{-\frac{1}{3}}.
\end{aligned} \tag{E-5}$$

The functions $s^{\frac{1}{2}}$, $t^{-\frac{1}{3}}$, $t^{\frac{1}{6}}$, $\ln t$ must be analytically continued appropriately.

Asymptotics of the period integrals

The asymptotic behaviour of the period integrals can be determined from the integral representation

$$-\frac{i}{\pi}\int_{e_i}^{e_j}\frac{x(3x^2-u)}{y}\mathrm{d}x, \tag{E-6}$$

with y given in (4.51), $y^2 = (x^3 - ux - v)^2 - \Lambda^6$, and e_i being the roots of y. We shall adopt the convention of [24] and choose ϕ_1 to be the integral around e_5,e_6, ϕ_2 around e_1,e_2, ϕ_{D1} around e_4, e_5, ϕ_{D2} around e_2, e_3. The roots of the elliptic curve are chosen so that in the limit $v \to 0$, $u \to \infty$ they give

$$\begin{aligned} e_1 &= -\sqrt{u} + \frac{1}{2}\frac{v}{u} - \frac{1}{2}\frac{\Lambda^3}{u} & e_2 &= -\sqrt{u} + \frac{1}{2}\frac{v}{u} + \frac{1}{2}\frac{\Lambda^3}{u} \\ e_3 &= -\frac{v}{u} - \frac{\Lambda^3}{u} & e_4 &= \frac{v}{u} - \frac{\Lambda^3}{u} \\ e_5 &= \sqrt{u} + \frac{1}{2}\frac{v}{u} - \frac{1}{2}\frac{\Lambda^3}{u} & e_6 &= \sqrt{u} + \frac{1}{2}\frac{v}{u} + \frac{1}{2}\frac{\Lambda^3}{u} \end{aligned} \tag{E-7}$$

In the following, $\alpha = v/u^{3/2}$ and $\beta = \Lambda^3/u^{3/2}$. The integrals of type α_i which give ϕ_i are easier. The integral around e_1, e_2 (i.e. ϕ_2) is

$$\phi_2 = -\frac{i}{\pi}\int_{e_1}^{e_2} \frac{x(3x^2-u)}{y}\mathrm{d}x. \tag{E-8}$$

This can be written by using the substitution $x = \sqrt{u}z$ as

$$\phi_2 = -\frac{i\sqrt{u}}{\pi}\int_{-1+\alpha/2-\beta/2}^{-1+\alpha/2+\beta/2} \frac{z(3z^2-1)}{\sqrt{(z^3-z-\alpha)^2-\beta^2}}\mathrm{d}x. \tag{E-9}$$

The integral itself is finite in the limit $\alpha \to 0$, $\beta \to 0$. Another substitution $z = -1 + \alpha/2 + s\beta/2$ and taking only the lowest order in α, β, gives us the final form

$$\phi_2 \approx \sqrt{u}(1-\alpha/2) = \sqrt{u} - \frac{v}{2u} \tag{E-10}$$

The dual integrals are slightly more complicated. The integral ϕ_{D2} around e_2, e_3 is after extraction of the factor $\sqrt{u}$,

$$\phi_{D2} = -\frac{i\sqrt{u}}{\pi}\int_{-1+\alpha/2+\beta/2}^{\alpha-\beta} \frac{z(3z^2-1)}{\sqrt{(z^3-z-\alpha)^2-\beta^2}}\mathrm{d}x. \tag{E-11}$$

This integral diverges logarithmically in the limit $\alpha \to 0$, $\beta \to 0$ as $\log(x+1)$, $x \to -1$. In order to also find the coefficients of the linear terms, we can cut the integral into two parts at a point γ. The integral over the interval $[\gamma, 0]$ will be finite, whereas the the integral over the interval $[-1+\alpha/2+\beta/2, \gamma]$ diverges logarithmically. We can simplify the term under the square root by writing as a product $\Pi_i(x - E_i)$, (E_i are rescaled roots $e_i = \sqrt{u}E_i$) and keep the α, β terms only in those roots that are close to our integration interval. The integral over the interval $[\gamma, \alpha - \beta]$ is finite, so we can drop all factors of α, β and find

$$I_1 = [3x + \log(x-1) - \log(x+1)]_\gamma^0$$

For the second integral we must keep the α, β terms only in the roots e_1, e_2

$$I_2 = \int_{-1+\alpha/2+\beta/2}^{\gamma} \frac{(3z^2-1)}{(z-1)\sqrt{(z+1-\alpha/2)^2-\beta^2/4}}\mathrm{d}x.$$

This integral can be calculated as

$$\begin{aligned}
I_2 = &\Big[3\sqrt{(x+1-\alpha/2)^2-\beta^2/4}+ \\
&+\frac{3}{2}\alpha\log\Big((2-\alpha+2x)+\sqrt{(x+1-\alpha/2)^2-\beta^2/4}\Big) \\
&-\frac{4}{\sqrt{16-\beta^2-\alpha^2-8\alpha}}\log\Big(8\sqrt{16-\beta^2-\alpha^2-8\alpha} \\
&\sqrt{(x+1-\alpha/2)^2-\beta^2/4}+32-2\beta^2+2\alpha^2-16\alpha+(16-4\alpha)(z-1)\Big)+ \\
&+\frac{4}{\sqrt{16-\beta^2-\alpha^2-8\alpha}}\log(x-1)\Big]_{-1+\alpha/2+\beta/2}^{\gamma}
\end{aligned}$$

The terms involving γ cancel as expected, and we are left with

$$\phi_{D2} = -\frac{i\sqrt{u}}{\pi}(3+\log\beta-3\log 2-\frac{3}{2}\alpha(\log\beta-\log 2)) \tag{E-12}$$

The behavior of ϕ_1 and ϕ_{D1} can be calculated in a similar way. The final asymptotic expansions of the period integrals are then

$$\begin{aligned}
\phi_1 &\approx \sqrt{u}+\frac{1}{2}\frac{v}{u} \qquad \phi_{D,1} \approx -\frac{i}{\pi}\left[\sqrt{u}\left(3+\frac{1}{2}\log\frac{\Lambda^6}{64u^3}\right)+\frac{3}{4}\frac{v}{u}\log\frac{\Lambda^6}{4u^3}\right] \\
\phi_2 &\approx \sqrt{u}-\frac{1}{2}\frac{v}{u} \qquad \phi_{D,2} \approx -\frac{i}{\pi}\left[\sqrt{u}\left(3+\frac{1}{2}\log\frac{\Lambda^6}{64u^3}\right)-\frac{3}{4}\frac{v}{u}\log\frac{\Lambda^6}{4u^3}\right]
\end{aligned} \tag{E-13}$$

Thus the period integrals can be found as linear combinations of the z_i's with the appropriate asymptotic behaviour

$$\begin{aligned}
\phi_1 &= \sqrt{3}\Lambda 2^{-\frac{1}{3}}z_3+\frac{2^{-\frac{1}{3}}\Lambda}{3}z_4 \\
\phi_2 &= \sqrt{3}\Lambda 2^{-\frac{1}{3}}z_3-\frac{2^{-\frac{1}{3}}\Lambda}{3}z_4 \\
\phi_{D,1} &= \sqrt{3}\Lambda 2^{-\frac{1}{3}}(-1)^{\frac{1}{3}}\left[\frac{1}{2\pi}\frac{\Gamma(-\frac{1}{6})\Gamma(\frac{5}{6})}{\Gamma(\frac{2}{3})}z_1+z_3\right]+ \\
&\quad+\frac{\Lambda 2^{\frac{2}{3}}}{3}(-1)^{-\frac{1}{6}}\left[-\frac{3}{4\pi}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}z_2+\frac{\sqrt{3}}{2}z_4\right] \\
\phi_{D,2} &= \sqrt{3}\Lambda 2^{-\frac{1}{3}}(-1)^{\frac{1}{3}}\left[\frac{1}{2\pi}\frac{\Gamma(-\frac{1}{6})\Gamma(\frac{5}{6})}{\Gamma(\frac{2}{3})}z_1+z_3\right]+ \\
&\quad-\frac{\Lambda 2^{\frac{2}{3}}}{3}(-1)^{-\frac{1}{6}}\left[-\frac{3}{4\pi}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}z_2+\frac{\sqrt{3}}{2}z_4\right].
\end{aligned} \tag{E-14}$$

First order derivatives

For further reference we will give here explicitly the first order derivatives of the fundamental solutions.

Just like the ordinary hypergeometric function, the derivative of the Appell function can be written in terms of an Appell function with different parameters (compare with (A-7))

$$\frac{\mathrm{d}}{\mathrm{d}x}F_4(a,b,c,c',x,y) = \frac{ab}{c}F(a+1,b+1,c+1,c',x,y) \tag{E-15}$$

and an analoguous relation for the y-derivative. Using this we can write the s,t derivatives of the fundamental solutions z_i as

$$\begin{aligned}
\frac{\partial z_1}{\partial s} &= \frac{1}{12}F_4\left(\frac{5}{6},\frac{5}{6},\frac{4}{3},\frac{2}{3},s,t\right)\\
\frac{\partial z_2}{\partial s} &= \frac{2}{27}s^{1/2}F_4\left(\frac{4}{3},\frac{4}{3},\frac{5}{2},\frac{2}{3},s,t\right)+\frac{1}{2}s^{-1/2}F_4\left(\frac{1}{3},\frac{1}{3},\frac{3}{2},\frac{2}{3},s,t\right)\\
\frac{\partial z_3}{\partial s} &= -\frac{1}{18}t^{-5/6}F_4\left(\frac{5}{6},\frac{7}{6},\frac{3}{2}.1,\frac{s}{t},\frac{1}{t}\right)\\
\frac{\partial z_4}{\partial s} &= \frac{1}{2}s^{-1/2}t^{-1/3}F_4\left(\frac{1}{3},\frac{2}{3},\frac{3}{2},1,\frac{s}{t},\frac{1}{t}\right)+s^{1/2}t^{-4/3}\frac{4}{27}F_4\left(\frac{4}{3},\frac{5}{3},\frac{5}{2},1,\frac{s}{t},\frac{1}{t}\right)\\
\frac{\partial z_1}{\partial t} &= \frac{1}{24}F_4\left(\frac{5}{6},\frac{5}{6},\frac{1}{3},\frac{5}{3},s,t\right)\\
\frac{\partial z_2}{\partial t} &= \frac{1}{6}s^{1/2}F_4\left(\frac{4}{3},\frac{4}{3},\frac{3}{2},\frac{5}{3},s,t\right)\\
\frac{\partial z_3}{\partial t} &= \frac{1}{16}t^{-5/6}F_4\left(-\frac{1}{6},\frac{1}{6},\frac{1}{2}.1,\frac{s}{t},\frac{1}{t}\right)+\frac{1}{18}t^{-11/6}sF_4\left(\frac{5}{6},\frac{7}{6},\frac{3}{2}.1,\frac{s}{t},\frac{1}{t}\right)+\\
&\quad+\frac{1}{36}t^{-11/6}F_4\left(\frac{5}{6},\frac{7}{6},\frac{1}{2},2,\frac{s}{t},\frac{1}{t}\right)\\
\frac{\partial z_4}{\partial t} &= -\frac{1}{3}s^{1/2}t^{-4/3}F_4\left(\frac{1}{3},\frac{2}{3},\frac{3}{2},1,\frac{s}{t},\frac{1}{t}\right)+s^{3/2}t^{-7/3}\frac{4}{27}F_4\left(\frac{4}{3},\frac{5}{3},\frac{5}{2},1,\frac{s}{t},\frac{1}{t}\right)-\\
&\quad-s^{1/2}t^{-7/3}\frac{2}{9}F_4\left(\frac{4}{3},\frac{5}{3},\frac{3}{2},2,\frac{s}{t},\frac{1}{t}\right).
\end{aligned} \tag{E-16}$$

The derivatives of the four periods ϕ_1, ϕ_2, ϕ_{D1}, ϕ_{D2} with respect to u and v can be easily, though tediously, found using the chain rule and (E-14). The derivatives of the period integrals are necessary for moduli space motion (6.15), and the period matrix τ ((4.49) and below)

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