Faculty of Science
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Diploma Thesis

Geometric phase in quantum theory

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I declare that I have worked out the diploma thesis independently and I have mentioned all literature sources I used.

Brno, January 2006

I would like to thank Tomas Tyc, M.S., PhD for leading of my diploma thesis, the provided literature, and the time he devoted to me.

Contents

Chapter 1

Introduction

1.1 A guide through this work

Although the concept of geometric phase came originally from the quantum theory, the similar phenomenon can be found also in the classical physics. In the beginning, I give an example of such the phenomenon, namely the one, arising from the motion of the globe. It should serve for having later a better insight into the original problem in quantum physics and, especially, one should see how it is closely related to the geometry.

Two different approaches to geometric phases are defined in the second chapter. The original Berry phase for the cyclic adiabatic evolutions and the Aharonov-Anandan phase for the general cyclic evolutions of the physical systems. The basic properties and connections between are briefly sketched and the formalism is illustrated on the well-known example, namely the occurance of an electron in a rotating magnetic field is discussed.

Some experiments manifesting the presence of a geometric phase factor (and important to my mind) are listed in the third chapter. Namely, the case of photons in a helically coiled optical fibre, the geometric phase in the Aharonov-Bohm effect and the geometric phase of three-level systems in interferometry is analyzed. To conclude this chapter, the (possible) applications are briefly discussed.

The rich mathematical tool is applied in the fifth chapter and the geometric phase is interpreted as arising from the holonomy in some bundle. This geometrical interpretation brings a new point of view. Especially the meaning of the adjective "geometrical" is clear, but moreover, the other properties of

the geometric phase are easy to be shown. The classification theorem is also briefly discussed. The adiabatic case is discussed in detail to get calculation useful formulae.

The last chapter is devoted to simple examples of the formalism introduced in the previous chapter.

1.2 Motivational example

Geometrical phases arise due to a phenomenon which can be described roughly as "a global change without any local change". For better understanding, the following example of the motion of the globe is especially illustrative.

We consider that the globe rotates with a constant angular velocity ω but that the direction of the vector of angular velocity $\vec{\omega}(t)$ varies in time. Moreover we assume that this change of the direction is slow with respect to ω. To be more precise, assume that the change is such that the vector of angular velocity rotates slowly with an angular velocity Ω around the z axis of a coordinate system, which is fixed in the outer space and which has the origin in the center of the globe. The vector $\vec{\omega}(t)$ remains in the xy-plane. Then we can write:

$$\vec{\omega}(t) = \omega \cdot (\cos \Omega t, \sin \Omega t, 0)^T.$$

We now choose a moving coordinate system $(\hat{x}, \hat{y}, \hat{z})$, in which the direction of $\omega(t)$ is fixed in $\hat{x}$-direction and the axes $\hat{y}$ and $\hat{z}$ remain "in the same direction", i.e. parallel, with respect to the globe. Rigorously, they are parallelly transported with respect to the globe, i.e. the infinitesimal change of the unit vectors $\vec{\hat{y}}$ and $\vec{\hat{z}}$ in every point is perpendicular to the globe. Then the equations of motion $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$ have a simple form

$$\frac{d}{dt} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} 0 & \Omega & 0 \\ -\Omega & 0 & -\omega \\ 0 & \omega & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}.$$

The matrix is obviously singular and has one real eigenvalue 0 with a corresponding eigenvector given by $c(t) \cdot (1, 0, \frac{\Omega}{\omega})^T$. This means nothing else but that in the so called adiabatic limit,i.e. $\frac{\Omega}{\omega} \to 0$, the point $(1, 0, 0)^T$, i.e. the north pole, is a stationary point. Thus we have proved that if the variation

of direction of angular velocity is small enough, the position of the rotation axis with respect to the body is fixed, which is referred to as an adiabatic theorem for the rigid body motion.

Let us now assume that the vector of angular velocity is slowly changing, for example in such a way as it is depicted in the picture 1.1. It means that in the beginning ($t = 0$) we have $(x, y, z) = (\hat{x}, \hat{y}, \hat{z})$. Then the vector $\vec{\omega}(t)$ moves along a meridian then along the equator and then along another meridian until it comes back to the starting point ($t = T$). According to the adiabatic theorem, the rotation axis traces the same closed path. It means that after the circuit, the globe will be in the same state $x = \hat{x}$ up to a rotation in the yz-plane. It is now easy to compute the angle of this rotation.

One could guess that the angle equals $\int_0^T \omega(t)dt$ at first glance. But a bit properer treatment shows that it is not the truth. It holds only in the coordinate system $(\hat{x}, \hat{y}, \hat{z})$. The angle of the rotation with respect to this system after the time T is really $\varphi_d = \int_0^T \omega(t)dt = \omega T$. But the system $(\hat{x}, \hat{y}, \hat{z})$ in the time $t = T$ does not coincide with the system (x, y, z), which is fixed in the outer space (in $t = 0$ they coincide). As the angular velocity is moving on the sphere, the unit vector $\vec{\hat{x}}$ is still normal to the sphere and the unit vectors $\vec{\hat{y}}, \vec{\hat{z}}$ are still the tangent vectors and they remain parallel (does not rotate around the $\hat{x}$ axis). In the words of differential geometry: they are parallelly transported along the closed path. Because of the curvature of the sphere (that represents the globe), there will be a nonzero angle φ_g between the axis $\hat{y}$ (or $\hat{z}$ equivalently) in $t = 0$ and $t = T$ (or equivalently between the vectors $\vec{\hat{y}}(t = 0)$ and $\vec{\hat{y}}(t = T)$). One can see this in figure 1.1. The result is: Although the axis of rotation comes back to the starting point after the time T, the globe is not in the same position. The final position differs from the original about the angle φ in the yz-plane which can be computed as the sum of two angles $\varphi = \varphi_d + \varphi_g$. I will call these angles phases in analogy with the quantum case. "Phase" is used meaning just "angle" for whatever possible argument of $\sin(\cdot)$ or $\cos(\cdot)$ or $\exp(\cdot)$. The first angle (phase) φ_d arise due to the angular velocity ω and the second angle (phase) φ_g arise due to the parallel transport on the sphere. Therefore the former can be called "dynamical phase" and the latter "geometric phase".

In the following, I will focus on the latter. The adjective "geometric" is apposite, because it is the geometry of the space on which the motion is fixed that determines the factor φ_g. Generally, as in our case of the

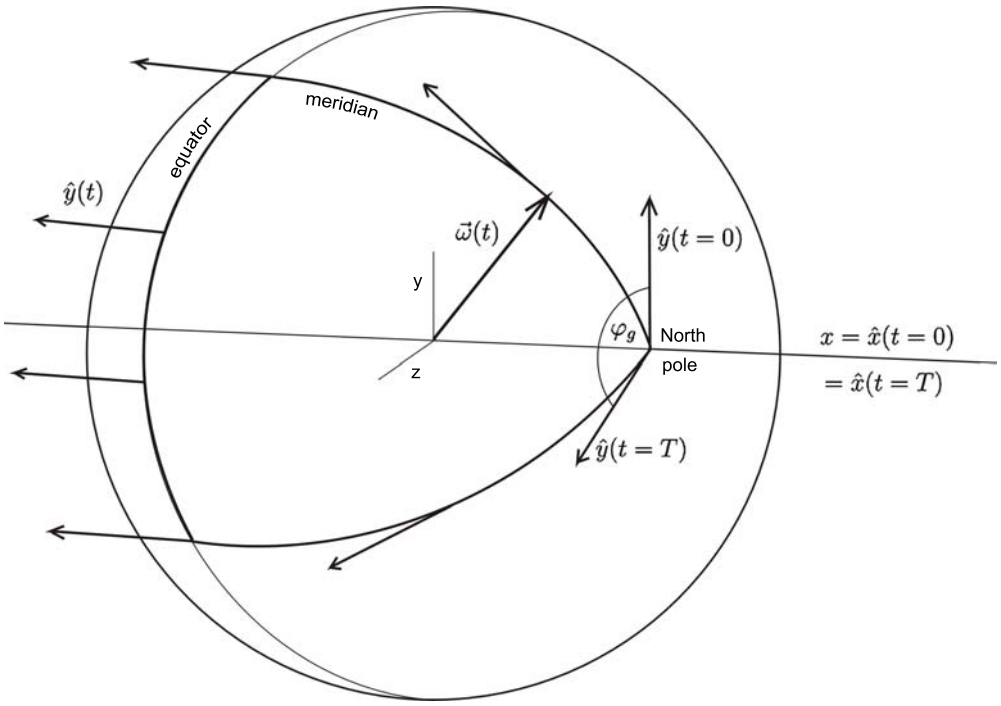


Figure 1.1: Parallel transport of the unit vector $\vec{\hat{y}}(t)$ in the direction of $\hat{y}$ axis

sphere, the curvature plays the crucial role. More precisely, the mapping $\vec{\hat{y}}(t=0) \to \vec{\hat{y}}(t=T)$ can be viewed as an element of the holonomy group of the Levi-Civita connection on the sphere (considered as the Riemannian manifold). We can then obtain the geometric phase φ_g as the integral of this connection along the closed path on the sphere. It can be transformed according to the Stoke's theorem to a surface integral of the differential of the connection, which is just the curvature of this connection in our case. We get consequently that the geometric phase φ_g in the case of the globe is proportional to the solid angle which determines the area on the sphere that is enclosed by the path traced by $\vec{\omega}(t)$ (point $(1,0,0)_{(\hat{x},\hat{y},\hat{z})}$).

Note that if we assumed a translation of the vector $\vec{\omega}(t)$ instead of the rotation ω, then such a translation of the globe would not produce any geometrical phase. The angle of rotation would be given only by φ_d in that case. This is because the motion of the point $(1,0,0)_{(\hat{x},\hat{y},\hat{z})}$ would be

fixed to the flat subspace. This is really the curvature of the sphere which is responsible for the geometric phase.

I would like to mention that the rotation of Foucault pendulum can also be explained by such a holonomy. In this example, the vector which determines the direction of oscillating and which is tangent to the sphere (Earth) is parallelly transported.

In these classical examples, "no local change" means that the tangent vector remains locally parallel during the whole evolution and "the global change" is the angle φ_g between the starting and the final vector. This can be explained by the holonomy of the natural connection on the tangent bundle of sphere.

Nearly the same situation occurs in quantum physics. Here a system picks up a geometric phase after a cyclic evolution. This is again given by the holonomy of a connection in a certain bundle.

Chapter 2

Geometric phases in physics

In this chapter, I introduce two different approaches to the so called geometric phase. The first part is dedicated the original derivation of Berry [2], which points out that a system which evolves cyclically under an adiabatic condition picks up an additional phase factor which turns out to be geometrical in nature. The Berry's concept is then shown on an example. A generalization to the degenerate case, done by Wilczek and Zee [5], is treated in the second part. Because of the special adiabatic condition, Aharonov and Anandan [3] tried to remove this condition and to generalize the occurance of the phase to evolutions that have to fulfill only the cyclic condition. This is introduced in the second part. Each part contains a general derivation of the ideas, I do not specialize in the geometrical meaning.

Before starting with the Berry phase, I should mention that, in fact, the first geometric phase was introduced by Pancharatnam already in 1956 [1]. In this article about the interference of polarized light, he defines a phase difference of two nonorthogonal polarization states. According to [1], two states are said to be "in phase" if the intensity of the superposed state is maximal. The phase difference is then the phase shift which has to be applied to one of the states in order to be in this relation with the second. Further, Pancharatnam points out that this phase has a geometrical meaning, which arises from the fact that the relation "to be in phase" is not transitive. Namely, let us consider three mutually nonorthogonal states of polarization, which are represented by three points A, B, C on the Poincaré sphere (Poincaré sphere is a well-known representation for the manifold of pure polarization states of a plane electromagnetic wave [1]). If the states are arranged such that A and B are "in phase" and B and C are "in phase",

then, in general, the states corresponding to the points A and C are not "'in phase". Pancharatnam also calculated the extend to which these last two states are "out of phase" and showed that this "phase difference" equals one half the solid angle on the Poincaré sphere determined by the spherical triangle ABC obtained by joining the vertices A,B and C by great circle arcs (geodesic arcs) on the sphere.

The results of Pancharatnam has been later interpreted in the context of all two-level quantum systems, for which the space of pure state density matrices is again the sphere S^2. The representation is determined by the obvious identification $SU(2)/U(1) = SO(3)/SO(2) = S^2$. For the three-level systems, there exists a generalization of the Poincaré sphere representation [4]. In that case, the coset space $SU(3)/U(2)$ is represented by a simply connected region in S^7.

2.1 Berry phase

In 1984, Berry published a paper [2] in which he considers cyclic evolutions of systems under special, so called adiabatic, conditions. He finds that a cyclic evolution of a wave function yields the original state plus a phase shift, and this phase shift is a sum of a dynamical phase and a geometric (or topological, or Berry) phase shift. Berry points out the geometrical character of this phase is not negligible. The phase is gauge invariant and therefore can not be gauged out as was earlier supposed. Many articles has been already written to this subject and, consequently, the so-called Berry phase is now well established, both theoretically as experimentally.

The original point of view of Berry is "dynamical". By this I mean that he starts with a Hamiltonian H that describes the quantum system in question. Further he considers that the Hamiltonian depends on a multidimensional real parameter x which parametrizes the environment of the system. Then the time evolution of the system is determined by the time dependent Schrödinger equation

$$H(x(t))|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$$

We can choose a basis of eigenstates $|n(x(t))\rangle$ corresponding to the energies E_n, i.e. such that

$$H(x(t))|n(x(t))\rangle = E_n(x(t))|n(x(t))\rangle$$

is fulfilled. In the moment, we assume that the energy spectrum of H is discrete, that the eigenvalues are not degenerated and that no level crossing occurs during the evolution. Moreover, suppose that the environment and therefore $x(t)$ is adiabatically varied. It means that the changes are slow in time with respect to the characteristic time scale of the system (given by the Planck's constant divided by the energy difference of two neighboring energy levels). Then the adiabatic theorem holds and thus, when the system starts in the n-th energy eigenstate, i.e. $|\psi(0)\rangle = |n(x(0))\rangle$, the system will be over the whole evolution at the n-th energy level. But, in general, the state vector gains a phase factor, i.e. $|\psi(t)\rangle = e^{i\phi_n}|n(x(t))\rangle$. At first sight, one would guess that the phase factor equals $\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(\tau)d\tau$ but the point is that the Schrödinger equation allows an additional phase factor $\gamma_n(t)$, i.e. $\phi_n = \theta_n + \gamma_n$. The former is now called the dynamical phase and the latter the geometric phase (notice the similarity to the motivational example). Putting this to the Schrödinger equation we get the following condition for γ_n:

$$\frac{\partial}{\partial t}|n(x)\rangle + i\frac{d}{dt}\gamma_n(t)|n(x)\rangle = 0$$

or equivalently in a nice form

$$\frac{d}{dt}\gamma_n(t) = i\langle n(x)|\frac{\partial}{\partial t}|n(x)\rangle = i\langle n|\nabla|n\rangle\frac{dx}{dt}.$$

Now, when we are given a cyclic evolution, described by a closed curve $C : t \mapsto x(t)$ with $x(T) = x(0)$, then the Berry phase for such an evolution is given by the following simple expression

$$\gamma_n(C) = i\oint_C \langle n(x)|\nabla|n(x)\rangle dx.$$

From this, one can easily see that the Berry phase depends on the geometry of the parameter space (and on the loop C therein). That is why Berry called this phase factor "geometric phase". Now, geometric phase is used as a universal notion for various generalizations of the original Berry's phase.

Let me briefly show how the Berry phase emerges in the concrete example. Namely, in the famous and important example when a spin-$\frac{1}{2}$ particle occurs in a magnetic field. I will proceed along the lines of [18]. Consider that the spin-$\frac{1}{2}$ particle is moving in an external magnetic field $\vec{B}$ which rotates adiabatically (slowly) under an angle θ around z-axis as it is depicted in 2.1. Then the magnetic field is given by

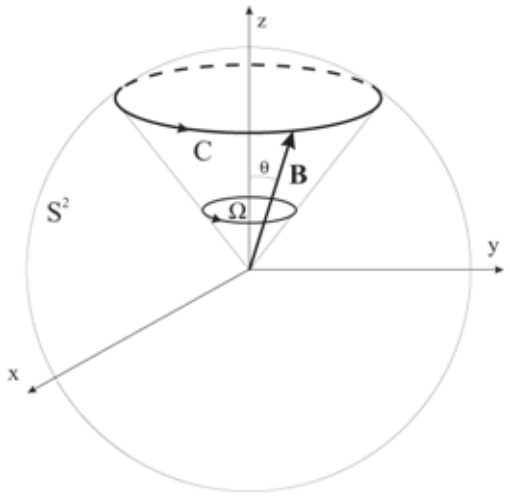


Figure 2.1:

$$\vec{B}(t) = B_0 \begin{pmatrix} \sin\theta\cos(\omega t) \\ \sin\theta\sin(\omega t) \\ \cos\theta \end{pmatrix}$$

where ω is the angular frequency of the rotation and $B_0 = |\vec{B}(t)|$. When the field rotates slowly enough and the expected value of the spin was in the direction of field, then the spin of the particle will follow the direction of the field and an eigenstate of the Hamiltonian $H(0)$ stays for all times t an eigenstate of $H(t)$. The interaction Hamiltonian for this system in the rest frame is given by $H(t) = \mu\vec{B}\cdot\vec{\sigma}$, where $\vec{\sigma}$ are Pauli matrices and $\mu = \frac{1}{2}\frac{e}{m}\hbar$ is the magnetic dipole moment connected with the spin. When we now use the explicit form of $\vec{B}$ we get two normalized eigenstates of $H(t)$

$$|n_+(t)\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\omega t}\sin\frac{\theta}{2} \end{pmatrix}, \quad |n_-(t)\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\omega t}\cos\frac{\theta}{2} \end{pmatrix}$$

with the corresponding energy eigenvalues $E_\pm = \pm\mu B_0$. A calculation of $\nabla|n_\pm(t)\rangle$ in the spherical coordinates $B_0, \theta, \phi(t) = \omega t$ leads to rather simple expressions

$$\langle n_+|\nabla|n_+\rangle = i\frac{\sin^2(\frac{\theta}{2})}{B_0\sin\theta}, \quad \langle n_-|\nabla|n_-\rangle = i\frac{\cos^2(\frac{\theta}{2})}{B_0\sin\theta}$$

The curve C in parameter space (which is now sphere) is given by $C : B_0 = const., \theta = const., \phi \in [0, 2\pi]$. Thus the Berry phase in this example equals

$$\gamma_{\pm}(C) = i \oint_C \langle n_{\pm}|\nabla|n_{\pm}\rangle B_0 \sin\theta d\phi = -\pi(1 \mp \cos\theta).$$

This is nothing else but the half of the solid angle enclosed by the path C and so we get the final expression

$$\gamma_{\pm}(C) = \mp\frac{1}{2}\Omega(C).$$

We see that whereas the dynamical phase (which is now given by $\pm\frac{\mu}{\hbar}B_0T$) depends on the period T of the rotation, the geometrical phase depends only on the special geometry of the problem.

2.2 Berry phase in the degenerate case

A generalization for the degenerate Hamiltonians was done by Wilczek and Zee [5]. From the comments above, one can deduce that the term $i\langle n(x)|\nabla|n(x)\rangle$ plays a role of a gauge potential in a $U(1)$ gauge field. This also shows that the Berry phase is a gauge invariant object and it is not possible to remove it by a certain choice of the basis states of the Hamiltonian. The direct generalization leads to non-abelian gauge field $U(n)$. Suppose that we are given a family of Hamiltonians $H(x)$ depending continuously on parameters x that has a n-times degenerate level for each value of x. By a simple renormalization of the energies, we can suppose that these levels are at $E = 0$. The degenerate levels are mapped back onto themselves by adiabatic development and this mapping is nontrivial in general. To show this, choose an arbitrary smooth set of bases $|n_a(t)\rangle$ for the various spaces of degenerate levels, so that

$$H(x(t))|n_a(t)\rangle = 0$$

Let the solutions $|m_a(t)\rangle$ of the Schrödinger equation with the initial condition $|m_a(0)\rangle = |n_a(0)\rangle$ are given by

$$|m_a(t)\rangle = U_{ab}(t)|n_b(t)\rangle.$$

Writing this equation, we have assumed the adiabatic evolution and our task is to determine $U(t)$. We demand that the $|m_a(t)\rangle$ remains normalized and this leads to the equation

$$(U^{-1}\dot{U})_{ab} = -\langle n_a|n_b\rangle \equiv A_{ab}$$

and this anti-Hermitian matrix A_{ab} plays the role of a gauge potential. Then the desired mapping given by a closed path in the parameter space has the form

$$U = Pe^{\oint A_\mu dx^\mu}$$

where P is the path ordering operator and x^μ are coordinates in the parameter space. This is known as the Wilson loop.

2.3 Aharonov-Anandan phase

In 1987, Aharonov and Anandan [3] considered cyclic evolutions that are not restricted by an adiabatic condition and purposed an generalization of Berry's phase. This generalization is very important, because the adiabatic condition is never exactly fulfilled for real evolutions. In the adiabatic approximation, Aharonov and Anandan phase then tends to the Berry phase if the parameters are chosen accordingly.

The appearance of Aharonov and Anandan phase (and other phases and related phenomena) can be explained in terms of quantum mechanics. The crucial role plays the fact that in quantum physics the physical state of a system is only determined up to a phase. The physical states are therefore in a bijective correspondence with points in the projective Hilbert space, i.e. with the pure-state density matrices. But usually, it is better to compute in Hilbert space and then pass to the projective space. This is due to the fact that the geometry of Hilbert space (and the computation therein) is simpler. The interplay between the Hilbert space and its projective space is responsible for the phenomena that I discuss here.

According to Aharonov and Anandan, I show the existence of a phase associated with cyclic evolution, which is universal in the sense that it is the same for the infinite number of possible motions along the curves in the Hilbert space $\mathcal{H}$ which project to a given closed curve in the projective Hilbert space $\mathcal{P}$ of rays of $\mathcal{H}$. Moreover, it is the same for all the possible Hamiltonians $H(t)$ which propagate the state along these curves.

The question is, what phase factor $e^{i\Phi}$ (which can have observable consequences) that the initial and the final state vector of a cyclic evolution may be related by. Suppose that the normalized state $|\psi(t)\rangle \in \mathcal{H}$ evolves according to the Schrödinger equation

$$H(t)|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle,$$

such that the evolution is cyclic, i.e. $|\psi(T)\rangle = e^{i\Phi}|\psi(0)\rangle$. Let $\Pi : \mathcal{H} \to \mathcal{P}$ be the projection map defined by

$$\Pi(|\psi\rangle) = \{|\psi'\rangle : |\psi'\rangle = c|\psi\rangle, c \text{ is a complex number}\}.$$

Then $|\psi(t)\rangle$ defines a curve $\tilde{C} : [0, T] \to \mathcal{H}$ with $C \equiv \Pi(\tilde{C})$ being a closed curve in $\mathcal{P}$. Conversely given any such curve $\tilde{C}$, we can define a Hamiltonian function $H(t)$ so that the Schrödinger equation is satisfied for the corresponding normalized $|\psi(t)\rangle$. Now define

$$|\tilde{\psi}(t)\rangle = e^{-if(t)}|\psi(t)\rangle$$

such that $f(T)-f(0) = \Phi$. Then $|\tilde{\psi}(t)\rangle$ is exactly cyclic, i.e. $|\tilde{\psi}(T)\rangle = |\tilde{\psi}(0)\rangle$, and from Schrödinger equation,

$$-\frac{df}{dt} = \frac{1}{\hbar}\langle\psi(t)|H|\psi(t)\rangle - \langle\tilde{\psi}(t)|i\frac{d}{dt}|\tilde{\psi}(t)\rangle.$$

Hence, if we remove the dynamical part from the phase Φ by defining

$$\beta \equiv \Phi + \frac{1}{\hbar}\int_0^T \langle\psi(t)|H|\psi(t)\rangle dt,$$

it follows from the above equation that

$$\beta = \int_0^T \langle\tilde{\psi}|i\frac{d}{dt}|\tilde{\psi}\rangle dt.$$

This is the final expression for the Aharonov-Anandan phase. Clearly, the same $|\tilde{\psi}(t)\rangle$ can be chosen for every curve $\tilde{C}$ for which $\Pi(\tilde{C}) = C$, by appropriate choice of $f(t)$. Hence such β, as defined above, is independent of the total phase Φ and Hamiltonian H for a given closed curve C. Indeed, from the last expression, β is independent of the parameter t of C, and is

uniquely defined up to $2\pi n$ (n = integer). Hence $e^{i\beta}$ is a geometric property of the unparameterized image of C in $\mathcal{P}$ only and therefore can be viewed as the second (or third) example of a geometric phase. Moreover, it is a universal phase in a certain sense (as we will see later).

Now, I will prove along the lines in [3] that in the adiabatic approximation, the phase found by Aharonov and Anandan tends to the phase found by Berry. Consider therefore a slowly varying Hamiltonian $H(t)$, with $H(t)|n(t)\rangle = E_n(t)|n(t)\rangle$, for a complete set $\{|n(t)\rangle\}$. If we set

$$|\psi(t)\rangle = \sum_n a_n(t) e^{-\frac{i}{\hbar}\int E_n dt}|n(t)\rangle,$$

then by using the Schrödinger equation and by differentiating the eigenvector equation, we obtain

$$\dot{a}_m = -a_m\langle m|\dot{m}\rangle - \sum_{n\neq m} a_n \frac{\langle m|\dot{H}|n\rangle}{E_n - E_m} e^{\frac{i}{\hbar}\int (E_m - E_n)dt},$$

where the dot denotes time derivative. In the adiabatic limit, we can further suppose that

$$\sum_{m\neq n}\left|\frac{\hbar\langle m|\dot{H}|n\rangle}{(E_n - E_m)^2}\right| \ll 1$$

holds and that the system starts in an eigenstate, i.e. $a_n(0) = \delta_{mn}$. Then the last term in the above expression of $\dot{a}_m$ is negligible and the system would therefore continue as an eigenstate of $H(t)$. In this approximation, we get

$$a_m(t) = e^{-\int\langle m|\dot{m}\rangle dt} a_m(0).$$

Thus for a cyclic adiabatic evolution, this yields the phase $i\int_0^T\langle m|\dot{m}\rangle dt$, which is independent of the chosen $|m(t)\rangle$ and which is the phase found by Berry.

Berry regarded this phase as a consequence of geometrical properties of the parameter space of which H is a function. But this phase is the same as Aharonov-Anandan phase β, when we approximate $|\tilde{\psi}(t)\rangle$ by $|m(t)\rangle$ and β, as defined, does not depend on any approximation and the expression $\beta = \int_0^T\langle\tilde{\psi}|i\frac{d}{dt}|\tilde{\psi}\rangle dt$ is exactly valid. Moreover, $|\psi(t)\rangle$ need not be an eigenstate of $H(t)$, unlike in the adiabatic case of Berry. It is neither necessary nor sufficient to go around a closed curve in parameter space in order to have a

cyclic evolution, with the associated geometric phase β. For these reasons, β is regarded as a geometric phase associated with a closed curve in the projective Hilbert space and not the parameter space, even in the special case considered by Berry. But given a cyclic evolution, a Hamiltonian $H(t)$ which generates this evolution can be found so that the adiabatic approximation is valid. Then β can be computed with the use of the expression given by Berry in terms of the eigenstates of this Hamiltonian.

Chapter 3

Experiments and applications

It seems to be impossible to measure the phase change $|\psi\rangle \to e^{i\varphi}|\psi\rangle$ at first sight, because the normalized vectors $|\psi\rangle$ and $e^{i\varphi}|\psi\rangle$ represent the same state of a physical system and hence the results of any measurement performed on them are the same. But, similarly to the wave optics, we can make some kind of an interference measurement. It means that some part of the physical system in question can serve as a reference phase. The other part on which the phase shift is performed is then recombined with the first one to form an interference pattern. This pattern is obviously different for different values of the phase shift φ.

The second task is to separate somehow the geometric phase from this total phase φ. Many of such experiments which measure the geometric phase shift have been already proposed and done. The Berry's phase can be demonstrated in experiments with photons by variation of the propagation direction. It is the case of the coiled optical fibre [6] or the Mach-Zehnder interferometer [7] for example. Other experiments with photons use variation of polarization to show up the Pancharatnam phase.

Another class of experiments are the experiments with neutrons. Neutrons are fermions which are not sensitive to any electric field and hence they are easy to handle. There are two groups of experiments with neutrons acquiring a geometric phase: neutron polarimeters and neutron interferometers. To the former, the experiment of Bitter and Dubbers belongs [8], where the effect of the Berry phase was first shown for fermions. To the latter, the experiment of Hasegawa, Zawisky, Rauch and Ioffe [9] belongs for example. Other relevant experiments are these of nuclear magnetic resonance, nuclear quadrupole resonance or atom interferometer.

3.1 Photons in an optical fibre

This was the first experiment to confirm the prediction of Berry. It was purposed by R.Y. Chiao and Y.S. Wu [10] in 1986 and this year yet realized by A. Tomita and R.Y. Chiao [6]. The photon's spin vector, which points either along the direction in which it is travelling or in the opposite direction, can be easily turned by changing the direction of travel. In [10] and [6], it is done with a coiled optical fibre. Let me introduce this experiment.

We assume that the light propagates inside the fibre in a single mode and its path is parametrized by the optical path length τ. Adiabatic condition is the conservation of the helicity which says, in other words, that at each point τ, the photon's spin state $|\vec{k}(\tau), \sigma\rangle$ satisfies

$$\vec{s} \cdot \vec{k}(\tau)|\vec{k}(\tau), \sigma\rangle = \sigma|\vec{k}(\tau), \sigma\rangle,$$

where $\vec{k}(\tau)$ is the direction of propagation of the photon at τ and $\sigma = \pm 1$ is its helicity quantum number. Formally, it is identical to the problem considered in the previous chapter for a spin $\vec{s}$ in an adiabatically changing magnetic field $\vec{B}(t)$, $g\vec{s} \cdot \vec{B}(t)|\vec{B}(t), m_s\rangle = E|\vec{B}(t), m_s\rangle$, where g is related to the gyromagnetic ratio and m_s is the component of the spin along the direction of $\vec{B}(t)$. Now we extend the results from this case to the case of photon.

Suppose that the fibre is wounded in such a way that the vector $\vec{k}$ traces out a closed curve, e.g. it is hellicaly shaped. In this case, we can use the derivation of Berry to determine the geometrical phase gained during the path through the fibre. Now, the parameter space is the momentum space $\{\vec{k}\}$ and the adiabatic invariant property is the helicity. Berry's formula for the photon is very similar to that for electrons. It only differs in a factor $\frac{1}{2}$ which is a consequence of the difference of spins of photons and electrons.

$$\gamma_\sigma = -\sigma\Omega(C).$$

The solid angle $\Omega(C)$ is determined by the curve C that the $\vec{k}$-vector traces out in momentum space (figure 3.1). Now, let us consider a linearly polarized light which is a superposition of the helicity eigenstates

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}(|k, +\rangle + |k, -\rangle).$$

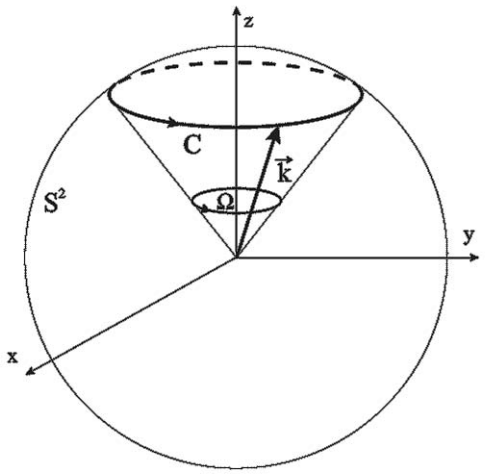


Figure 3.1: The solid angle Ω in momentum space.

One easily shows that the final state, after the propagation through the helix, is then given by (if we ignore for the moment the dynamical phase factor)

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}\left(e^{i\gamma_+}|k,+\rangle + e^{-i\gamma_+}|k,-\rangle\right).$$

With this equation we can compute the following squared transition amplitude

$$|\langle\psi_i|\psi_f\rangle|^2 = \cos^2(\gamma_+).$$

This can be interpreted after Malus law as a rotation of the plane of polarization about an angle of γ_+. That means the optical fibre, wound into a helix shaped form, leads to an effective optical activity although the material of the fibre has no optical active characteristics. The amount of the rotation indeed does not depend on the wavelength of the light but on the solid angle and therefore it is a pure geometrical effect.

In the particular experiment of Tomita and Chiao [6], they use approximately 2 meters long single-mode fibre. The fibre is helically wound onto a cylinder, which can be seen in figure 3.2. The ends of the fibre point into the same direction to ensure the closed path in $\vec{k}$-space. The polarization of the light coming from a He-Ne laser is controlled by polarizers as well as the polarization of the light leaving the fibre.

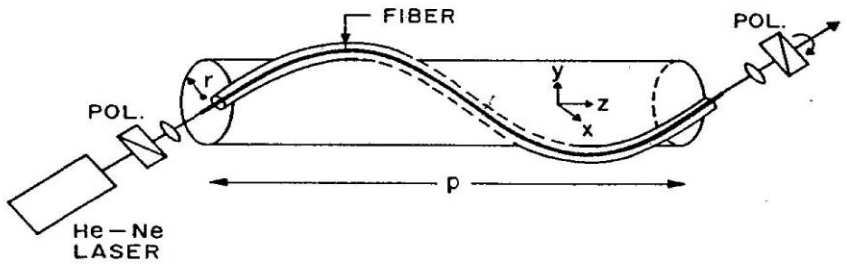


Figure 3.2: Experimental setup for measuring of the Berry's phase in a helical optical fibre [6]

The experiment consists of two major parts. The first one is to use a fibre witch is wound with a constant pitch angle θ. The Berry formula reads as

$$\gamma_\sigma(C) = -2\pi\sigma(1-\cos\theta) = -2\pi\sigma(1-\frac{p}{s}),$$

where s is the length of the fibre and p is the length of the cylinder. This equation was experimentally verified.

For the second part of the experiment, they used nonuniform wound helical fibres, i.e. with the pitch angle θ dependent on a parameter τ. Then the solid angle of the closed curve C, traced out in momentum space, is given by

$$\Omega(C) = \int_0^{2\pi} (1-\cos\theta(\tau))\, d\tau$$

and the Berry phase is $\gamma_\sigma = -\sigma\Omega(C)$, which is again related to an optical rotation which is measured. The measurements then verify these theoretical predictions.

Note that these optical effects could be explained in principle entirely classically in terms of Maxwell equations with the appropriate boundary conditions. When the mutually orthogonal triad of vectors $\vec{k}, \vec{E}$ and $\vec{B}$ will adiabatically propagate by parallel transport inside a wound fibre, it would lead to the above results. The problem is in that the classical theory fails for low photon number, whereas the quantum theory still holds. Fundamentally, it is the the Bose nature of the photon which allows the appearance of these

optical manifestations of Berry phase on a classical level. These effects can be therefore considered as topological features of classical Maxwell theory which arise in quantum mechanics, but survive the correspondence-principle limit ($\hbar \to 0$) into the classical level.

3.2 Geometric phase and Aharonov-Bohm effect

In 1959, Y. Aharonov and D. Bohm demonstrated that the vector potential has more physical significance that had been previously thought [11]. They send two beams of electrons past a long tightly wound solenoid along both sides (as in 3.3 depicted). It is well-known that the magnetic field of the

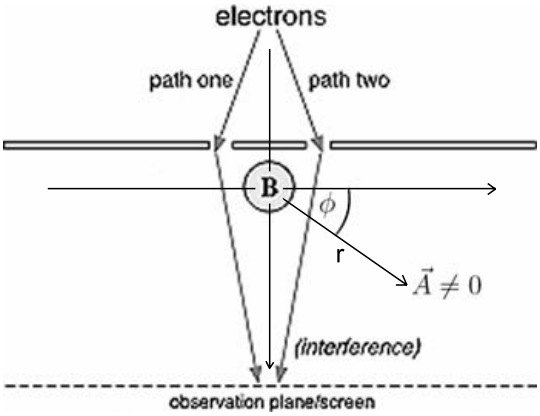


Figure 3.3: Measurement of the Aharonov-Bohm effect.

solenoid is very simple: it is uniform inside (and parallel to) the solenoid, and zero outside. Although the electrons occur only in the region where $B = 0$, the change of interference pattern, observed when the beams are recombined, manifests a phase difference of electrons. This phase difference arise due to the key fact that the vector potential A is nonzero outside the solenoid.

When we choose a Coulomb gauge (i.e. $\nabla \cdot A = 0$), the vector potential outside the solenoid equals $A = \frac{\Phi}{2\pi r}\hat{\phi}$, where Φ is the magnetic flux, and r and $\hat{\phi}$ (the azimuthal unit vector) are defined by setting the axis of

the solenoid as the axis of a cylindrical coordinate system (see figure 3.3). Aharonov and Bohm realized that the presence of the nonzero vector potential fundamentally changed the behavior of the wave function. Space, for the wave function, is no longer simply connected and the integrals $\oint A \cdot dr$ are path dependent. It is precisely the difference of such integrals between the two different paths around the solenoid that give rise to the Aharonov-Bohm effect observed as a shift of the interference fringes.

Now, following [2], I show how this effect may be seen in terms of a geometric phase. The splitting and recombination of the beam of electrons can be viewed in such a way that the electrons goes backwards in time along one path and returns along the other path to its original state at the same time. It defines a circuit C around the solenoid. Let the physical (quantal) system, described by $|n(R)\rangle$, consist of an electron (or electrons), with a charge e, which is in a box that is centered in R and lies outside the solenoid. In the case of $A = 0$, the Hamiltonian for an electron depends on the position of electron $\hat{r}$ and has a form of $H(\hat{p}, \hat{r} - R)$. The corresponding wavefunctions are $\psi_n(r - R)$ with energies E_n independent of R. Now, when $A \neq 0$, the states $|n(R)\rangle$ satisfy the eigenequation

$$H\left(\hat{p} - eA(\hat{r}), \hat{r} - R\right)|n(R)\rangle = E_n|n(R)\rangle.$$

One can verify that the exact solutions (in the coordinate representation) of this equation are obtained by multiplying ψ_n by an appropriate phase factor as follows:

$$\langle r|n(R)\rangle = e^{\frac{ie}{\hbar}\int_R^r A(r')\cdot dr'}\psi_n(r - R).$$

Since the solenoid is not inside the box, the integral in this equation is independent of the path from R to r. When we transport the system (box) round the circuit C, the system $|n(R)\rangle$ acquires a geometric phase factor. This factor can be calculated using the Berry's formula from previous chapter and using

$$\begin{aligned}\langle n(R)|\nabla_R n(R)\rangle &= \int\int\int d^3r\bar{\psi}_n(r - R)\left(-\tfrac{ie}{\hbar}A(R)\psi_n(r - R) + \nabla_R\psi_n(r - R)\right)\\ &= -\tfrac{ie}{\hbar}A(R),\end{aligned}$$

where the simplification is a consequence of the normalization of the wavefunctions ψ_n. And finally, the Berry phase is given by

$$\gamma_n(C) = \frac{e}{\hbar}\oint_C A(R)\cdot dR = \frac{e\Phi}{\hbar}.$$

The final line follows from the Stokes theorem. This result agrees with that obtained in [11]. It is independent of n and also of C if this winds once and only once round the solenoid. It is also obvious that C can be taken such that R is two-dimensional.

It means that the parameter space of the Hamiltonian H is in fact $\mathbb{R}^2$. But this space is locally flat against the previous example, where the parameter space was a sphere S^2, which is obviously curved. One can not in any case use the Berry's solid angle formula and it seems so that the geometric phase should be zero, although it is not the truth as we have computed. This "mystery" is the same as the "mystery" of the Aharonov-Bohm effect-the electron occurs in a field with $B = 0$ but nevertheless, it gains a phase factor. The core of this "mystery" is in that the Hamiltonian has a singular point in $r = 0$. Thus the parameter space is $\mathbb{R}^2 \setminus \{0\}$ which is not simply connected and the path integrals $\oint A(R) \cdot dR$ occuring in the expression of the geometric phase are nonzero for the paths which enclose the singularity and hence the geometric phase is nonzero. There is no problem to consider $R \in \mathbb{R}^3$. In this case, the singularity becomes a singular line l (representing the solenoid) and thus the parameter space is $\mathbb{R}^3 \setminus \{l\}$. The solenoid even need not to be long tight but can have the shape of a torus T. In this case, the parameter space would be $\mathbb{R}^3 \setminus \{T\}$ and the geometric phase would be the same of course. This is due to the fact that the geometric phase depends on the fundamental group of parameter space only and $\pi_1(\mathbb{R}^2 \setminus \{0\}) = \pi_1(\mathbb{R}^3 \setminus \{l\}) = \pi_1(\mathbb{R}^3 \setminus \{T\})$. In this example, the topology (rather than geometry) of the parameter space plays the crucial role.

3.3 Three-level systems in interferometery

The two preceding examples give a possibility of measurement of the geometric phase for a photon and for an electron respectively. These examples are rather different in that the geometric phase arises in a different way, as I noted. But in both cases, it is the (abelian) Berry phase. We can view this obviously as the (universal) Aharonov-Anandan phase. It means to consider this geometric phase only as a functional of the closed curve in appropriate (projective) Hilbert space and do not matter what Hamiltonian produced the curve. Such a purely kinematic derivation is done in [25]. In that point of view, both of the foregoing geometric phases arise in the evolution of $U(1)$-invariant states - the states of the Poincaré sphere ($S^2 \cong SU(2)/U(1)$) and

states of $\mathbb{R}^2$ respectively.

Now, I will briefly discuss the example of a three-level system, where the states are $U(2)$-invariant. An example of such a system can be the system of three photons $|\psi\rangle$. A general $SU(3)$ transformation is then realized by a three-level interferometer. It is a sequence of several beam splitters. Since every beam splitter corresponds to a $SU(2)$ transformation, such a construction is possible (every element of $SU(3)$ can be obviously decomposed as a product of several elements of $SU_{ij}(2)$, $i, j \in \{1, 2, 3\}$). The $SU(3)$ transformations that produce the same physical state form a group that is isomorphic to $U(2)$. This group can be obtained as the stability group of $|\psi\rangle$ up to a phase [21]. Therefore, the space of states can be identified with $SU(3)/U(2)$ and the states are obviously $U(2)$-invariant.

When we use the purely kinematical approach of [25], the geometric phase associated with a, generally open, curve C in $SU(3)/U(2)$ is given as follows

$$\varphi_g[C](=\beta) = \varphi_{tot}[\tilde{C}] - \varphi_{dyn}[\tilde{C}],$$

where, of course, the total phase and the dynamical phase are defined to be $\varphi_{tot}[\tilde{C}] := \arg\langle\psi(s_1)|\psi(s_2)\rangle$ and $\varphi_{dyn}[\tilde{C}] := \mathrm{Im}\int_{s_1}^{s_2} ds\langle\psi(s), \dot{\psi}(s)\rangle$ respectively. The $\tilde{C} \in SU(3)$ is an arbitrary lift of the curve C and $s_1 \leq s \leq s_2$ is its parametrization. It can be shown that two points in the state space $SU(3)/U(2)$ can be connected by a unique arc for which the geometric phase is zero. Such a curve is called a geodetic arc [25]. Now, consider (for example) three arbitrary state vectors $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ which we connect by geodetic arcs. It turns out that the geometric phase associated with such geodetic triangle is given simply by $\varphi_g = \langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_3\rangle\langle\psi_3|\psi_1\rangle$, which is known as the Bargmann invariant.

A nice experiment, manifesting a geometric phase of a system of three photons, was proposed in [12]. Therein, an optical scheme is introduced to produce and detect an abelian geometric phase shift which arises from such transformation along a geodesic triangle. The scheme employs a three-channel optical interferometer and four experimentally adjustable parameters. The $SU(3)$ transformation is realized by a sequence of unitary transformations given by optical elements inside the three-channel interferometer. The space of output states of the interferometer can be identified with $SU(3)/U(2)$. This space is a generalization of the Poincaré sphere [4].

Such experiment is particularly interesting, because there is a big difference against the two previous experiments, where the curves were generated by a Hamiltonian and the parameter was time t. In this case, the

curve is parametrized by an evolution parameter s, which is a function of the adjustable parameters of the interferometer. We can adjust these parameters such that the output state evolves cyclically along the geodesic triangle $\psi^{(1)} \to \psi^{(2)} \to \psi^{(3)} \to e^{i\varphi_g}\psi^{(1)}$ in the (four-dimensional) state space $SU(3)/U(2)$ (figure 3.4). The geometric phase φ_g associated to this trian-

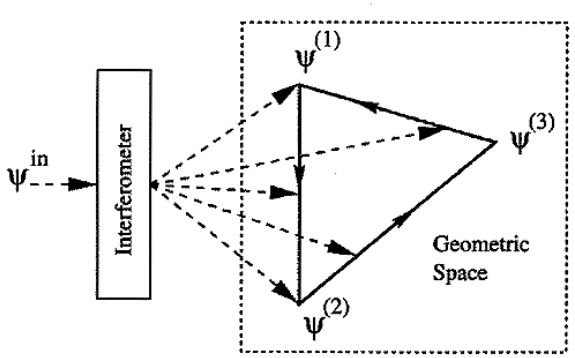


Figure 3.4: By adjusting the parameters of the interferometer, the output state in the geometric space can be made to evolve along geodesic paths, from one vertex to the next, until the triangle is closed.

gle is given by the appropriate Bargmann invariant. Thus, it can be easily computed as a function of adjustable parameters of interferometer ([12]).

A key technical challenge is measuring of the geometric phase φ_g, because one must have a reference state with which to interfere the output state. The input state is bad choice because it contains a dynamical phase due to evolution through the interferometer. However, this optical phase can be eliminated through the use of a counterpropagating beam. The concrete experimental setup is described in [12]. Two beams, orthogonally polarized, are propagating through the interferometer at the same time in the opposite directions and in the output they interfere. The interference pattern then gives the relative phase difference $2\varphi_g$, which should confirm the theoretical prediction.

3.4 Applications

As I have already mentioned, the experiment with a coiled optical fibre can be understood with classical Maxwell equations. The other two experiments can be also, in fact, explained by a classical physics. Nevertheless, in 1991 Kwiat and Chiao [13] did an experiment with entangled photons, where the quantal character of the phase was confirmed.

By Schrödinger, entanglement was denoted as a fundamental concept of quantum mechanics. The approach of Berry phase can by applied to the entanglement and then various Bell inequalities are established. These are inequalities with various expectation values, which have to be satisfied by every local realistic theory. But the Bell inequalities can be violated by quantum mechanics. Also, Bell inequalities that involve Berry phases exist and they are also violated by quantum mechanics, which again demonstrates quantum nature of the geometric phase.

It is useful to note that the geometric phase, e.g. in the Aharon-Bohm effect, can be an exploited for extremely precise measurements of field characteristics, e.g. magnetic flux, via detecting interference fringes shifts.

The most important (to my mind) application of the geometric phase is in quantum computation. The unit of quantum information is called qubit (quantum bit) and is realized by a quantum system with two accessible orthogonal eigenstates represented by the two boolean values $|0\rangle$ and $|1\rangle$. But in contrast to a classical bit, this system can also exist in any superposition $\alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$. This is one of the reasons why quantum computers are more powerful than classical computers. A quantum logic gate is then a device that performs a unitary operation on a qubit. A general operation is possible to realize by using one and two-qubit operations, e.g. by Hadamard gate, phase gate, C-NOT gate and controlled phase shift gate [18].

Geometric phases seem to be good candidates for realizing low noise quantum computing devices. Because of the dependence only on the geometry of the appropriate space the geometric phase is an ideal construction for fault-tolerant quantum computation. Nevertheless, it has also several drawbacks, e.g. one has to get rid of the dynamical phase. There are many physical realizations of quantum computers and quantum gates. For example, the optical photon quantum computer, where the polarization states of the photon represents the two base states. Other example of a suitable physical systems is nuclear magnetic resonance (see for instance [26]).

Chapter 4

Geometrical interpretation

In the case of the geometric phase in quantum mechanics, the situation is similar to that in the case of the motivational example in the beginning of this work. It turns out that the geometric phase is given by a holonomy in a certain principal fibre bundle. Then its geometric nature and many properties become obvious. It brings a new point of view in which the problem can be well understand without much computation. Last but not least, it gives a nice application of a rich mathematical theory.

4.1 Holonomy interpretations of the geometric phase

A short time after Berry had found the geometrical phase factor ([2]), Barry Simon in [20] argued that it is precisely the holonomy in a Hermitian line bundle since the adiabatic theorem naturally defines a connection in such a bundle. This not only took the mystery out of Berry's phase factor, but provided calculational simple formulas. Let us now discuss this Simon's nondegenerate case in more detail.

In the definition of the Berry phase in chapter 2, we considered a Hermitian operator (Hamiltonian of the system) $H(x)$ depending smoothly on a parameter $x \in M$, with an isolated nondegenerate eigenvalue $E_n(x)$ depending continuously on x. Let me restrict to the unitary evolution for the moment. Then the eigenstates $|n(x)\rangle$ of $H(x)$ are assumed to be normalized and it is straightforward that the association $x \mapsto |n(x)\rangle$ defines a principal

fibre bundle λ over the parameter space M with fibres

$$L_x = \{|\psi_x\rangle : |\psi_x\rangle = e^{i\varphi}|n(x)\rangle, \varphi \in \mathbb{R}\} \cong U(1).$$

The point is that twisting of this $U(1)$-principal bundle $U(1) \to \lambda \to M$ affects the phase of quantum mechanical wave functions.

Consider a closed curve $C : [0,T] \ni t \to x(t) \in M$ in the parameter space ($x(0) = x(T)$) and set $H(t) := H(x(t))$. When the system evolves adiabatically along this loop, the state vector $|\psi(t)\rangle$ of the system which is initially an eigenstate $|\psi(0)\rangle := |n(0)\rangle$ of the Hamiltonian $H(0)$ evolves according to the Schrödinger equation and remains always an eigenstate of $H(t)$ and therefore, after time T, gains a phase factor

$$|\psi(T)\rangle = e^{-\frac{i}{\hbar}\int_0^T E_n(t)dt}|n(T)\rangle = e^{-\frac{i}{\hbar}\int_0^T E_n(t)dt}e^{i\gamma(T)}|\psi(0)\rangle$$

as explained in the chapter 2. The Berry's additional phase factor $\gamma(t)$ obviously defines a way of transporting of the basis $|n(t)\rangle$ along C, i.e. a lift $\tilde{C} \in \lambda$ of the loop $C \in M$, by associating $t \mapsto |n(t)\rangle = \gamma(t)|n(0)\rangle$, i.e. a connection ω in the principal fibre bundle. According to the chapter 2, we know that the local expression of the connection (i.e. the gauge potential) is given simply by the $U(1)$-valued one-form

$$A := -\langle n(x)|d|n(x)\rangle = -\langle n(x)|\frac{\partial}{\partial x^\mu}|n(x)\rangle dx^\mu.$$

In this Berry-Simon (adiabatic) approach (further BS approach), the term $e^{i\gamma(C)} = e^{\oint_C A}$ is thus nothing else but the element of the local holonomy group $\mathrm{Hol}_{|n(0)\rangle}(\omega)$ based at the point $|n(0)\rangle \in \lambda$ coming from the loop C. Of course, such a holonomy group is (in general) a subgroup of the structure group $U(1)$ of λ. Note that in this nondegenerate case the Lie algebra of the structure group is one-dimensional and therefore $A \wedge A = 0$ and we can use the Stoke's theorem to compute the Berry's phase directly, as an integral of the curvature ρ of the connection ω: $i\gamma(C) = \oint_C A = \int\int_S dA = \int\int_S(\rho - A \wedge A) = \int\int_S \rho$, where $S = \partial C$.

Note also that if we consider a general, nonunitary, evolution we get an equivalent description. Instead of λ we have an associated complex line bundle $L = \lambda \times_{U(1)} \mathbb{C}$ with fibres isomorphic to $\mathbb{C}$. The expression of the gauge potential is the same (but the form is $\mathbb{C}$-valued now) and $e^{i\gamma_n(C)}$ is again an element of the holonomy group of this bundle, which is a (sub)group of $GL(L_x) \cong U(1)$.

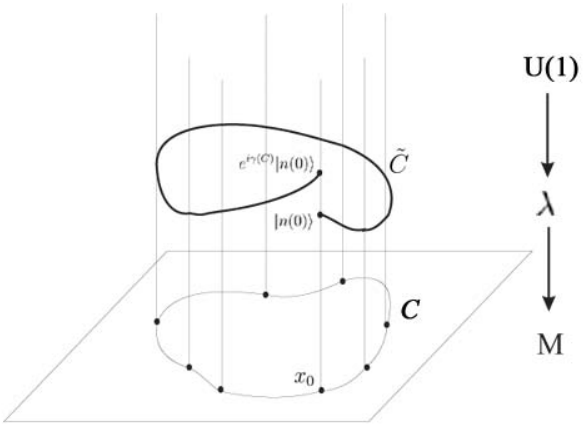


Figure 4.1: The fibre bundle for the Berry phase

As I described in the chapter 2, Aharonov and Anandan [3] (and Anandan and Stodolsky [14]), realized that instead of considering a closed loop in parameter space, one could consider the closed loop in state space (or equivalently projective Hilbert space) and drop the adiabaticity condition. In the Aharonov-Anandan (further AA) approach, one can again restrict to the unitary case and consider a $U(1)$-principal bundle η, over the state space (projective Hilbert space) $\mathcal{P} = \mathbb{C}P^N$ $(N = dim(\mathcal{H}) - 1)$ with fibres

$$\eta_{|\psi\rangle\langle\psi|} = \left\{e^{i\delta}|\psi\rangle, e^{i\delta} \in U(1)\right\} \cong U(1)$$

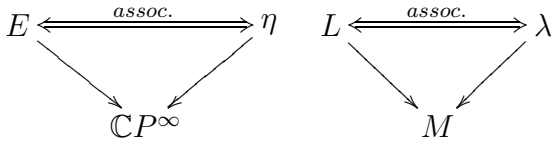
over a point $|\psi\rangle\langle\psi|$ in $\mathcal{P}$. The connection in this AA bundle $\eta : U(1) \to \mathcal{H}^* \to \mathcal{P}$ is given in a quite natural way - the horizontal subspaces are perpendicular to the fibres with respect to the Hilbert space inner product. In the local form this reads as

$$\mathcal{A} = -\langle\psi|d|\psi\rangle,$$

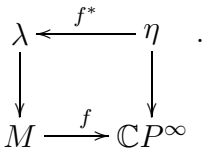
which is again a $U(1)$-valued one-form. Then the Aharon-Anandan's geometric phase is given by $\beta = -i \oint_C \mathcal{A}$ for a curve C in $\mathcal{P}$. Note that the tangent vectors remains normalized which says the real part of $\langle\psi|\frac{d}{dt}|\psi\rangle$ is identically zero and thus the bundle is, in fact, a subbundle $S^1 \to S^{2N-1} \to \mathbb{C}P^{N-1}$ and the connection therein is the common one. The $e^{i\beta} = e^{\oint \mathcal{A}}$ is then the holonomy associated with this connection and it is again an element of $U(1)$.

Again, we can consider the nonunitary evolutions which are now described by curves in the associated complex line bundle $E = \eta \times_{U(1)} \mathbb{C}$. The expressions for the gauge potential and geometric phase are obviously the same.

Considering a quite general case we can take an inductive limit, i.e. to allow $N \to \infty$. Then, actually, $\mathbb{C}P^\infty = \bigcup_N \mathbb{C}P^N$ and we can put $\mathcal{P} = \mathbb{C}P^\infty$. Now, we can depict the situation we have as follows:



As I have shown in the chapter 2, the Aharonov-Anandan phase tends to the Berry phase in the adiabatic approximation. Thus the geometrical interpretations in AA approach and BS approach have to be somehow linked. To see the relation between the bundles λ, η, notice that η is the universal classifying bundle for $U(1)$ principal fibre bundles [15]. Then, according to the classifying theorem for such bundles, the desired relation becomes obvious. The theorem says that any $U(1)$ principal fibre bundle λ over M is isomorphic to the pull-back bundle $f^*(\eta)$ for some continuous function $f : M \to \mathbb{C}P^\infty$, i.e. the following diagram is commutative



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Moreover, the bundle $\lambda \cong f^*(\eta)$ depends only on the homotopy class $[f] \in [M, \mathbb{C}P^\infty]$ of f, i.e. homotopic maps induce isomorphic bundles if M is paracompact. It means in turn that it is the topology of the manifold M which determines all possible $U(1)$ bundles over M. For instance, if $f : M \to \mathbb{C}P^\infty$ is homotopic to a constant map, then one will obviously obtain the trivial bundle $\lambda \cong f^*(\eta) \cong M \times U(1)$. I would like to point out that the triviality of the bundle does not necessarily imply that the holonomy group is trivial and hence the geometric phase in this bundle can be nonzero.

The same commutative diagram can be drawn also for the associated line bundles L and E. E is the universal classifying space for complex line bundles and thus the bundle λ is obtained as the pullback of f and its topology is determined by the homotopy class of f.

To classify all principal (or line) bundles with a given base manifold M, we need to know more about the structure of $[M, \mathbb{C}P^\infty]$. From an elementary algebraic topology, we know the homotopy structure of $\mathbb{C}P^\infty$, namely $\pi_i(\mathbb{C}P^\infty) = \pi_{i-1}(S^1)$ holds and it means that the space $\mathbb{C}P^\infty$ has only the second homotopy group nontrivial (it equals $\mathbb{Z}$) and thus is the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$ [15]. For such a space we can use the Hurewicz theorem ([15]) which gives the correspondence with homology groups. Finally, we obtain the relation $[M, \mathbb{C}P^\infty] = [M, K(\mathbb{Z}, 2)] = H^2(M; \mathbb{Z})$. In other words, the principal (line) bundles, which form a group with respect to the $U(1)$ (tensor) product, are (as groups) isomorphic to the second cohomology group $H^2(M; \mathbb{Z})$ (if M is "normal" = if it has a homotopy type of a CW complex; see chapter 4 in [15]). The isomorphism is given by the first Chern number $c_1(M) = f^*(c_1)$, where $c_1 \in H^1(\eta, \mathbb{Z})$ ([**?**]). In such a way we have classified all $U(1)$ (complex line) bundles.

The explicit form of the map $f : M \to \mathbb{C}P^\infty$ needed for computing the geometric phase is obviously fixed by the given Hamiltonian, namely,

$$f : M \ni x \mapsto f(x) = |n(x)\rangle\langle n(x)| \in \mathbb{C}P^\infty,$$

where $|n(x)\rangle$ is normalized and $H(x)|n(x)\rangle = E_n(x)|n(x)\rangle$ holds. For such f, we obtain the BS bundle λ as the pullback bundle $f^*(\eta)$ (and also $L = f^*(\eta)$ holds). Moreover, the natural AA connection, as defined earlier, is really universal in the sense that the adiabatic connection A in the BS parameter space bundle is obtainable as the pullback of the universal AA connection $\mathcal{A}$, i.e. $A = f^*(\mathcal{A})$.

When we are given a system with the family of Hamiltonians which commute with a time reversal operator T, and if in addition $T^2 = +1$, then the Hilbert space can be taken over the real numbers. Then the situation becomes complete analogous. One only has to replace the complex projective space $\mathbb{C}P^\infty$ by real projective space $\mathbb{R}P^\infty$, Eilenberg-MacLane space $K(\mathbb{Z}, 2)$ by $K(\mathbb{Z}_2, 1)$ and Chern class $c_1(E) \in H^2(M; \mathbb{Z})$ by Stiefel-Whitney class $w_1(E) \in H^1(M; \mathbb{Z}_2)$. The corresponding bundles have $O(1) = \mathbb{Z}_2$ as their structure group. In this case the holonomy does not come from a connection, because the fibres are discrete and thus the parallel transport is unique. Another, interesting, situation appears when the time reversal operator T fulfills $T^2 = -1$ (the case of fermionic time reversal invariant systems - see [16]). It gives to the Hilbert space a quaternionic structure. In that case, the AA bundle is $Sp(1) \to S^\infty \to \mathbb{H}P^\infty$ with the structure being $Sp(1) \cong SU(2)$,

the group of unit quaternions. The first nontrivial homotopy group of the projective quaternionic space is $\pi_4(\mathbb{H}P^\infty) = \pi_3(S^3) = \mathbb{Z}$. However, also the higher homotopy groups are nontrivial and hence $\mathbb{H}P^\infty$ is not an Eilenberg-MacLane space. In this case, The maps $f : M \to \mathbb{H}P^\infty$ induce $SP(1)$-bundles over M which have well-defined second Chern class (it is an element of $H^4(M, \mathbb{Z})$), but they do not classify all the $Sp(1)$-bundles. That is the difference against the complex (and real) case.

4.2 Degenerate case

In comparison to the previous section, I will now discuss the case that the Hamiltonian $H(x)$, which depends on the multidimensional parameter $x \in M$, has degenerate eigenvalues. Suppose that $E_n(x)$, the nth eigenvalue of $H(x)$ is $\mathcal{N}$-fold degenerate and that no level crossing occurs. Obviously, the homotopy type of M is (in general) nontrivial. The eigenspaces, corresponding to $E_n(x)$ are $\mathcal{N}$-dimensional and we can pick the single valued frame, i.e. the orthonormal basis $\{|n_i(x)\rangle; i = 1, ..., \mathcal{N}\}$. They are transformed into each other by a $U(\mathcal{N})$ transformations, which can be viewed as the gauge transformations. The suitable mathematical framework for this case is given by the $U(\mathcal{N})$ principal fibre bundle over M. It is a straightforward generalization of the nondegenerate case from the preceding section. Now, in the BS picture, we have a bundle

$$U(\mathcal{N}) \to \lambda_{\mathcal{N}} \to M$$

According to the chapter 2, in this BS (adiabatic) approach, the connection in this bundle (viewed as a $\mathfrak{u}(\mathcal{N})$-valued form) locally reads as

$$A_{ij}(x) = -\langle n_i(x)|d|n_j(x)\rangle = -\langle n_i(x)|\frac{\partial}{\partial x^\mu}|n_j(x)\rangle dx^\mu.$$

The nonabelian phase factor picked up by system when going adiabatically along a lop C in the parameter space M is then given by a Wilson loop $U_{ij} = Pe^{\oint_C A_{ij}}$. This can be again viewed as an element of the holonomy group of the bundle $\lambda_{\mathcal{N}}$ which is now a (sub)group of $U(\mathcal{N})$. In the case of the real Hilbert space, the BS bunle has a structure group $O(\mathcal{N})$.

The generalization of the AA approach to the degenerate case goes as follows. The desired principal $U(\mathcal{N})$ bundle is now

$$\eta_{\mathcal{N}} : U(\mathcal{N}) \to V_{\mathcal{N}}(\mathbb{C}^\infty) \to G_{\mathcal{N}}(\mathbb{C}^\infty),$$

where $V_{\mathcal{N}}(\mathbb{C}^\infty)$ is the Stiefel manifold, the space of $\mathcal{N}$-frames in $\mathbb{C}^\infty$. This is topologized as a subspace of the product of $\mathcal{N}$ copies of the unit sphere in $\mathbb{C}^\infty$. $G_{\mathcal{N}}(\mathbb{C}^\infty)$ is the Grassman manifold, the space of $\mathcal{N}$-dimensional vector subspaces of $\mathbb{C}^\infty$. It is topologized as a quotient space via the natural projection $V_{\mathcal{N}}(\mathbb{C}^\infty) \to G_{\mathcal{N}}(\mathbb{C}^\infty)$.

An equivalent way, how one can see this, is to consider $G_{\mathcal{N}}(\mathbb{C}^\infty)$ as the space of $\mathcal{N}$-dimensional (orthogonal) projection operators Λ, i.e. self-adjoint operators $\Lambda^* = \Lambda$ which fulfill $\Lambda^2 = \Lambda$, $Tr\Lambda = \mathcal{N}$. The space $V_{\mathcal{N}}(\mathbb{C}^\infty)$ is then the space of partial isometries, which are operators ν with the property that $\nu\nu^* = \Lambda$. The canonical projection $\pi_{\mathcal{N}} : V_{\mathcal{N}}(\mathbb{C}^\infty) \to G_{\mathcal{N}}(\mathbb{C}^\infty)$ is given by $\pi_{\mathcal{N}}(\nu) = \nu\nu^* = \Lambda$.

The AA bundle possesses a natural connection, it is the Stiefel connection, which can be in the above notation written in the form $\tau = -\nu^* d\nu$. If we choose a local section of $\eta_{\mathcal{N}}$, which is the same as a choice of the frame $\{|\psi_i(\Lambda)\rangle; i = 1, ..., \mathcal{N}\}$, the connection (viewed as the $\mathfrak{u}(\mathcal{N})$-valued one-form) has the local description:

$$\mathcal{A}_{ij} = -\langle\psi_i|d|\psi_j\rangle$$

The nonabelian phase factor (holonomy) picked up during the evolution described by a loop in $G_{\mathcal{N}}(\mathbb{C}^\infty)$ is then given by $\mathcal{U}_{ij} = Pe^{\oint_C \mathcal{A}_{ij}}$.

Similarly to the $\mathcal{N} = 1$ case, the bundle $\eta_{\mathcal{N}}$ is the universal classifying bundle of $U(\mathcal{N})$ principal fibre bundles. It means in turn that each BS bundle $\lambda_{\mathcal{N}}$ over a parameter space M can be obtained as the pullback bundle $f^*(\eta_{\mathcal{N}})$ of the AA bundle $\eta_{\mathcal{N}}$, i.e. the following diagram is commutative:

$$\begin{array}{ccc} \lambda_{\mathcal{N}} & \xleftarrow{f^*} & \eta_{\mathcal{N}} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & G_{\mathcal{N}}(\mathbb{C}^\infty) \end{array}$$

Thus the $U(\mathcal{N})$ bundles are determined by the homotopy class $[M, G_{\mathcal{N}}(\mathbb{C}^\infty)]$. Similarly, one can consider the real case with $O(\mathcal{N})$ and (with some constraints) the quaternionic case with $Sp(\mathcal{N})$ as the structure group.

For such a degenerate adiabatic case, the map f is again determined by the Hamiltonian $H(x)$. It associates to every $x \in M$ an eigenprojector $|\psi_x\rangle\langle\psi_x| = \Lambda(x) \in G_{\mathcal{N}}(\mathbb{C}^\infty)$, where $H(x)\Lambda(x) = \Lambda(x)H(x) = E_n(x)\Lambda(x)$. The connection in the BS bundle $\lambda_{\mathcal{N}}$ is then given by the pullback of the Stiefel connection in $\eta_{\mathcal{N}}$, i.e. $A_{ij} = f^*(\mathcal{A}_{ij})$.

4.3 Structure of the parameter space

Till now, I have not said anything about what the space of parameters M might be. It is important both for the classification, i.e. what the space $[M, \mathbb{C}P^\infty]$ (or alternatively $H^2(M;\mathbb{Z})$) looks like, and also for the computation of the geometric phase.

Let us consider that the Hamiltonian describing the system is of the form

$$H(x) = \epsilon \sum_{i=1}^{N^2} x^i J_i,$$

where $(x^i) \in \mathbb{R}^{N^2} \setminus \{0\}$, ϵ is a constant with the dimension of energy and, finally, J_i are generators of a compact semisimple Lie group G. Every Hamiltonian describing an N-level system, which can be obviously viewed as an element of the vector space of $N \times N$ Hermitian matrices, can be written in such a "linear" form. Moreover, since also every J_i has to be Hermitian, the Hamiltonian $H(x)$ can be regarded as an element of the Lie algebra $\mathfrak{u}(N)$ of the Lie group $U(N)$. Therefore, the Hilbert space possesses a unitary representation of the group G and the example of $G = U(N)$ plays a universal role.

According to Jordan decomposition ([19]), there exists always an unitary operator $U(x)$ that diagonalize the Hamiltonian $H(x)$, i.e.

$$U^\dagger(x)H(x)U(x) = H_D = \mathrm{diag}(E_1(x), ..., E_N(x))$$

for the eigenvalues $E_1(x), ..., E_N(x)$ of the Hamiltonian. Such operators $U(x)$ are clearly not uniquely defined. I show (along the lines of [17]) that the parameter space for the Hamiltonian $H(x) = U(x)H_D U^\dagger(x)$ is a submanifold of a flag manifold G/T, where T is the maximal torus.

Since G is assumed to be semisimple, we can choose a Cartan subalgebra $\mathfrak{h}$ of a complexification of its Lie algebra $\mathfrak{g}_\mathbb{C}$ and consider a corresponding root decomposition

$$\mathfrak{g}_\mathbb{C} = \mathfrak{h} \oplus_\alpha \mathfrak{g}_\alpha.$$

Furthermore, G is assumed to be compact and hence its Lie algebra $\mathfrak{g}$ can be viewed as the (unique) compact real form of $\mathfrak{g}$ ([19]). Then we know that $\mathfrak{g}$ splits as:

$$\mathfrak{g} = i\mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Delta^+} \left(\mathbb{R}(X_\alpha - X_{-\alpha}) + i\mathbb{R}(X_\alpha + X_{-\alpha})\right).$$

The elements $X_\alpha \in \mathfrak{g}_\alpha$ can be chosen such that $[X_\alpha, X_{-\alpha}] = H_\alpha$, where H_α lies in real Cartan subalgebra $\mathfrak{h}_0$, and, for every two roots α, β, the following holds. $[X_\alpha, X_\beta] = 0$ if $\alpha + \beta$ is not a root and $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$ for some $N_{\alpha,\beta} \in \mathbb{R}$ with $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. The equation above says that each element X of the Lie algebra $\mathfrak{g}$ can be expressed as:

$$X = i \sum_{i=1}^{l} y^i H_i + i \sum_{\alpha \in \Delta^+} (z^\alpha X_\alpha + \bar{z}^\alpha X_{-\alpha})$$

for some choice of the coefficients y^i, z^α (l is the rank of G = dimension of $\mathfrak{h}_0$ = number of nodes in corresponding Dynkin diagramm [19]). Consequently, using the exponential map $\mathfrak{g} \to G$, which is diffeomorphism on the connected component of identity, $U(x)$ can be written as:

$$U(x) = e^{X(t)} = e^{i \sum_{\alpha \in \Delta^+} (z^\alpha(x) X_\alpha + \bar{z}^\alpha(x) X_{-\alpha})} e^{i \sum_{i=1}^{l} y^i(x) H_i}.$$

Since $H_D(x)$ is diagonal, it belongs to the Cartan subalgebra and we may write $H_D(x) = \sum_{i=1}^{l} E_i(x) H_i$. Since the generators H_i of the Cartan subalgebra commute, we can use the two last equations to simplify the expression of $H(x)$:

$$H(x) = e^{i \sum_{\alpha \in \Delta^+} (z^\alpha(x) X_\alpha + \bar{z}^\alpha(x) X_{-\alpha})} H_D(x) e^{-i \sum_{\alpha \in \Delta^+} (z^\alpha(x) X_\alpha + \bar{z}^\alpha(x) X_{-\alpha})}.$$

Since $H_D(x)$ plays no role in the expression for geometric phase (in fact can be chosen to be constant), this proves that the geometric phase is locally given by the coefficients z^α only, which obviously corresponds to the coordinates of the flag manifold G/T, i.e. the last expression proves that the actual parameter space M is a submanifold of G/T. Obviously, G/T is naturally embedded in $\mathbb{R}^{N^2}$ as the orbit of adjoint action of G on a regular element $H \in i\mathfrak{h}_0$.

In fact, we can proceed further, similarly to [17], and to specify M more. We can consider that the eigenvalues of Hamiltonian is constant. The eigenstates $|n(x)\rangle$ of H_D then fulfill the time independent eigenvalue equation $H_D(x)|n(x)\rangle = E_n|n(x)\rangle$ and it means, in turn, that the eigenvectors $|n(x)\rangle$ are precisely the weight vectors of the present representation of G on the Hilbert space $\mathcal{H}$. Now, let us define a map $\Psi : G_{\mathbb{C}} \to \mathcal{P}$ for a fixed vector $|\psi\rangle \in \mathcal{H}$ by $\Psi(g) := [U(g)|\psi\rangle]$, where $U(g)$ is the representation of the complexified $G_{\mathbb{C}}$ and $[U(g)|\psi\rangle]$ denotes the ray through $U(g)|\psi\rangle$. This map is

obviously not bijective. But, we can factorize Ψ to a bijective $\hat{\Psi} : G_{\mathbb{C}}/P \to \mathcal{P}$, where

$$P := \{g \in G_{\mathbb{C}} : U(g)|\psi\rangle = c|\psi\rangle, c \in \mathbb{C}\}\,.$$

If we choose the vector $|\psi\rangle$ to be the highest weight vector of the representation, we see that the Borel subgroup B is a subgroup of P(i.e. P is parabolic). Therefore, $G_{\mathbb{C}}/P$ is a compact submanifold of $G_{\mathbb{C}}/B$. But, the homogeneous space $G_{\mathbb{C}}/B$ is diffeomorphic to the flag manifold G/T and thus the parameter space M is a homogeneous space $G_{\mathbb{C}}/P \subset G/T$ in general. If it is a proper subgroup or not, it depends on the representation of G on $\mathcal{H}$. Namely, it depends on the position of the highest weight $\Lambda \in \mathfrak{h}^*$, which uniquely determines the irreducible part of the present representation, in the Weyl chamber. M is a proper subgroup of G/T iff Λ lies on a wall of Weyl chamber [19]. One can observe that the map f which maps M to the projective Hilbert space $\mathcal{P}$ is given exactly by $\hat{\Psi}$.

For example, consider the group $G = SU(N+1)$ in the standard representation on the Hilbert space $\mathcal{H}$. The standard representation is itself a fundamental representation and therefore its highest weight Λ lies on the wall. It is easy to see, that the group P, as defined above, equals $U(N)$ and thus the parameter space is $M = SU(N+1)/U(N) = \mathbb{C}P^N = \mathcal{P}$. When we take $SU(3)$ as the group G and consider its octet representation ([19]), then the highest weight (which is now a sum of two fundamental weights) lies in the interior of Weyl chamber and thus $P = B$ in this case. The parameter space is then $M = SU(3)/U(1) \times U(1)$ and the map f maps M into $\mathcal{P} = \mathbb{C}P^7$.

4.4 More on the Berry's phase

Now, when we know that the Hamiltonian is (usually) parametrized by the points in a homogeneous space G/P, we can use the structure of the homogeneous space to simplify the formulas for Berry phase. Such cases are discussed in [22] and, in fact, in [21] where a different, "kinematical", approach is used.

When the Hamiltonian is diagonalized by $U(x)$ as before, the evolution of instantaneous basis (with respect to which the system gains only the dynamical phase) is governed by this operator. Then, treating all eigenstates simultaneously, the Berry phase is given by the, so called, Berry matrix $U^\dagger(x)dU(x)$. Namely, for the n-th eigenstate and starting basis $\{|n_i(0)\rangle\}$ the gauge potential is $A_{ij} = \langle n_i(0)|U^\dagger(x)dU(x)|n_j(0)\rangle$. Hence Berry matrix is a

$\mathfrak{g}$-valued one-form and, by projecting on a certain eigenspace, determines the connection needed for computation of Berry phase. We may write the Berry matrix in terms of the generators of G. Denoting S_i the generators of P and T_a the generators of G/P, the Cartan forms η^i and ω^a are defined by

$$U^\dagger(x)dU(x) = \eta^k(x)S_k + \omega^a(x)T_a,$$

where $\eta^k(x) = \eta^k_\mu(x)dx^\mu$ and $\omega^a(x) = \omega^a_\mu(x)dx^\mu$. x^μ are, as usual, the coordinates in $M = G/P$. Now, the gauge potential for the n-th eigenspace is given by

$$A_{ij} = \eta^k_\mu(x)\langle n_i(0)|S_k|n_j(0)\rangle + \omega^a_\mu(x)\langle n_i(0)|T_a|n_j(0)\rangle.$$

The Berry phase is then obtained as the curve integral of this expression, in which we have separated the terms that are fixed by the particular representation from the terms that are to be integrated and depends on the geometry of parameter space. Other additional group structures and properties of particular representation can be use for further simplifications. A detailed discussion, both of a general case and also of many examples, is in [21]. Of course, the Berry connection A_{ij} relates to the natural Riemannian connection on the homogeneous space G/P. In fact, the BS bundle is a homogeneous vector bundle and the relation is known [29], but I do not intend to discuss this here.

4.5 The non-adiabatic case

Suppose again that we are given a manifold M of the parameters x, on which the Hamiltonian $H(x)$ of the system continuously depends and which represents the change of the environment. Suppose further that the environment, and hence the state of the system, will somehow vary and after a time T, the environment and also the state of the system will be the same as in the beginning. It defines a closed curve $C = x(t)$ in the space M of parameters. Consider that the system was initially in a state described by a vector $|\psi_{x(0)}\rangle$ in an appropriate Hilbert space $\mathcal{H}$. Then the evolution of the system can be described by a family of state vectors $|\psi_{x(t)}\rangle$ which solve the Schrödinger eguation. In this way, the Schrödinger equation together with the loop C defines a curve $\tilde{C}$ in the Hilbert space $\mathcal{H}$. Due to the cyclic condition, the final state vector $|\psi_{x(T)}\rangle$ differs from the initial $|\psi_{x(0)}\rangle$ about a phase factor

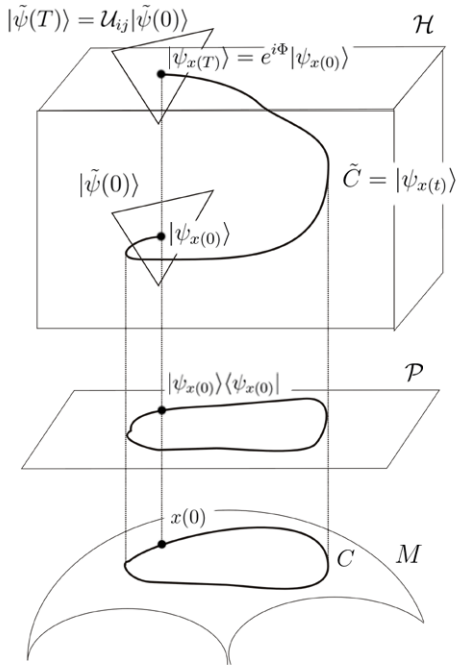

Figure 4.2: Aharonov-Anandan geometric phase corresponding to a curve $\tilde{C}$ generated by a general cyclic evolution of parameters C.

Φ and thus it defines a closed curve in a projective space $\mathcal{P}$ and also in the Grassman manifold $G_{\mathcal{N}}$. It is visualized in figure 4.2.

In this case, the Aharonov-Anandan's geometric phase is well defined. In any point $\psi_{x(t)}$ we can choose a frame (orthonormal basis) $\{|\psi_i(t)\rangle; i = 1, ..., \mathcal{N}\}$ for a suitable $\mathcal{N}$. It defines a section in a Stiefel bundle $V_{\mathcal{N}}$ over a Grassman manifold $G_{\mathcal{N}}$. There we have a canonical connection defined in such a way that the horizontal subspaces are orthogonal to the fibres. The nonabelian geometric phase factor is then given as a holonomy of this connection. Notice the similarity to the motivational example. An orthonormal basis $\{|\tilde{\psi}_i(0)\rangle\} = \{|\psi_i(0)\rangle\}$ is parallelly transported along a curve $\tilde{C}$ to the point $|\psi_{x(T)}\rangle$, where $|\tilde{\psi}_i(T)\rangle = \sum_j \mathcal{U}_{ij}|\tilde{\psi}_j(0)\rangle$. With respect to such a parallelly transported basis, the system gains only the dynamical phase

factor. The nonabelian geometric phase factor (holonomy transformation) $\mathcal{U}_{ij}$ is given by the parallel transport.

But, we do not assume an adiabatic change of the parameter x and thus the frames $\{|\psi_i(t)\rangle\}$ is not posibble to choose being the eigenstates of the Hamiltonians $H(x)$ in general. Such an exact cyclic evolution (i.e. corresponding to a closed curve in the projective Hilbert space) allows in general only the universal AA approach to the geometric phase. In the case of a nonadiabatic evolution, we cannot define any BS bundle and we cannot use the Berry's adiabatic treatment. Although AA approach is very useful theoretically, the BS approach is preferable for computation. The geometric phase, in the BS approach, is identified with the associated holonomy of the loops in the space of parameters and it means that one does not need to solve the time dependent Schrödinger equation.

Moreover, it is not complicated to show ([24]) that an evolution given by Schrödinger equation can never be exactly described by any eigenprojector $\Lambda(x)$, i.e. the frames $\{|\psi_i(t)\rangle\}$ are never simultaneous eigenvectors of $H(x)$. It means that the adiabatic condition is never exactly fulfilled and it can be only a good approximation in some cases. Therefore it would be useful to develope a Berry-like treatment for the geometric phase also for nonadiabatic evolutions, at least for some of them.

Suppose accordingly that we have such a physical system that the two following conditions are satisfied:

1. The cyclic states are the eigenstates of a Hermitian operator $\tilde{H} = \tilde{H}(x)$.

2. $\tilde{H}$ is related to the Hamiltonian according to $\tilde{H}(x) = H(F(x))$, for a diffeomorphism $F : M \to M$.

For this class of quantum systems, one can still use the classification theorem. This is realized by replacing f by a map $\tilde{f}$ defined by $\tilde{f} = f \circ F : M \to G_{\mathcal{N}}$. Then we obtain the (nonadiabatic) bundle $\tilde{\lambda}_{\mathcal{N}}$ over the parameter space M as the pullback bundle $\tilde{\lambda}_{\mathcal{N}} = \tilde{f}^*(\eta_{\mathcal{N}})$. The map $\tilde{f}$ also pullbacks the universal connection $\mathcal{A}_{ij}$ and yields a nonadiabatic Berry connection $\tilde{A}_{ij} = \tilde{f}^*(\mathcal{A}_{ij})$ on $\tilde{\lambda}_{\mathcal{N}}$. The geometric phase is then obtainable as the holonomy of this connection

$$\mathcal{U}_{ij} = Pe^{\oint \mathcal{A}_{ij}} = Pe^{\oint_C \tilde{A}_{ij}(x)} = Pe^{-\oint_C \langle n_i(F(x))|\frac{\partial}{\partial x^\mu}|n_j(F(x))\rangle dx^\mu}.$$

In the adiabatic limit, F approaches to the identity map. Hence, F lies in the connected component of $Diff(M)$ and thus is homotopic to the identity map. Therefore $[f] = [\tilde{f}]$ and the bundles $\lambda_{\mathcal{N}}$ and $\tilde{\lambda}_{\mathcal{N}}$ have the same topology.

The conditions 1. and 2. seem to be quite restrictive. But, it turns out that the first condition is fulfilled for any periodic Hamiltonian [17]. Neither the necessary nor the sufficient conditions for the existence of F for general cyclic Hamiltonians are known, but there are examples where F exists and our analysis applies, e.g. the example of cranked Hamiltonian.

Chapter 5

Simple examples

In this section, I show how the formalism derivated in the previous chapters applies to simple examples. These, related to the adiabatic case are treated mainly according to [22], [5] and [20]. In the nonadiabatic case, I follow the lines of [27] and [28].

5.1 The adiabatic nondegenerate case

Let me review the example of spin-$\frac{1}{2}$ particle in an external magnetic field. Assume that the Hamiltonian is given simply by $H(\mathbf{x}) = \mathbf{x} \cdot \mathbf{S}$, where $\mathbf{x} \in \mathbb{R} \setminus \{0\}$ and $\mathbf{S}$ is a spin-$\frac{1}{2}$ operator ($[S_i, S_j] = i\epsilon_{ijk}S_k$) on $\mathbb{C}^2$. The actual group G is obviously $SU(2)$ and its representation on the Hilbert space is the standard one. Thus the Hamiltonian can be parametrized by $SU(2)/U(1)$. The Hamiltonian is already in the linear form and thus we know that can be diagonalized by an operator $U(\mathbf{x})$ that is given by $e^{-i\mathbf{z}\cdot\mathbf{S}}$, where $z_3 = 0$, i.e. $U(\mathbf{x})$ is not generated by any element from the Cartan subalgebra. Our task is to determine this operator, i.e. $\mathbf{z}$. One can show that $\mathbf{z} \perp \mathbf{x}$ has to hold and, finally, $|\mathbf{z}| = 1$. The energy eigenvalues are precisely the eigenvalues s of the element S_3 of the Cartan subalgebra, which are weights of the present, standard, representation. These are easy to be computed: $s = -\frac{1}{2}, 0, \frac{1}{2}$. The corresponding bundles are given by associating

$$\mathbf{x} \mapsto |s(\mathbf{x})\rangle := U(\mathbf{x})|s\rangle.$$

The connection in this bundle is given by

$$A_s = -\langle s|d(-i\mathbf{z}\cdot\mathbf{S})|s\rangle = -i\langle s|d(z_1)S_1 + d(z_2)S_2|s\rangle.$$

I did not separate the terms coming from representation and the geometrical terms, because in such a nondegenerate case, we can use the Stoke's theorem to proceed further. Using a spherical coordinate system one shows that $F_s = s\mathcal{S}(\mathbf{x})$, where $\mathcal{S}(\mathbf{x})$ is the ares form on the sphere in $\mathbf{x}$. It means that at the energy level $E = 0$, the Berry phase vanishes. For the energies $E = \pm\frac{1}{2}$, it is given by the solid angle formula $\gamma_{\pm\frac{1}{2}} = \pm\Omega(C)$.

5.2 The adiabatic degenerate case

An another nice example is the system described by the Hamiltonian $H(x) = x_i\epsilon_{ijk}\sigma_j \otimes \sigma_k$. Using, de facto, the same approach, one can compute the eigenvalues to be $-2, 0, 2$ and a further investigation reveals that the Berry phase vanishes for $E = \pm 2$. For the eigenvalue $E = 0$, which is two-degenerate, one obtains the Berry phase as a two times two diagonal matrix that has precisely the terms $\pm\Omega(C)$ on the diagonal [23].

A simple example of a nonabelian Berry phase is introduced in [5]. The Hamiltonian is considered to be given by $H = R(t)H_D R^{-1}(t)$, where H_D is a constant diagonal matrix with nth times degenerate eigenvalue $E = 0$ and $R(t) := R(\theta(t))$ is the $SO(n+1)$ rotation, i.e.

$$R(\theta) = e^{i\theta_n T_{n,n+1}} \cdots e^{i\theta_2 T_{2,n+1} e^{i\theta_1 T_{1,n+1}}},$$

where $T_{i,n+1}$ are the generators of $SO(n+1)/SO(n)$. One of the intuitive explanation of the phenomenon of geometric phase can be such that the embedding of the relevant subgroup P into G varies in time, which can be seen on this example. The parameter space is, of course, the sphere $S^n = SO(n+1)/SO(n)$ and the operator responsible for the geometric phase is $R(\theta)$ now. The gauge potential is

$$A_{ij} = -\langle n_i(\theta)|d|n_j(\theta)\rangle$$

which is now a $\mathfrak{so}(n)$-valued form. For $n = 3$, we can find explicitly

$$A = \sin\theta_1 T_{12} d\theta_2 + (\sin\theta_1 \cos\theta_2 T_{13} + \sin\theta_2 T_{23})\, d\theta_3.$$

The berry phase is then, as usual, obtained as the integral $U_{ij} = -i \oint_C A_{ij}$. A wide range of interesting adiabatic examples can be found and easily computed, see for instance [**?**].

5.3 The nonadiabatic case

Let us consider again a quantum system whose Hamiltonian is a function of generators of a compact semisimple Lie group G. Then, as I explained in the previous chapter, every Hamiltonian, which in the linear representation reads $H_0 = \beta \cdot \mathbf{X}$, can be expressed as follows

$$H_0 = \epsilon e^{i\sum_{\alpha\in\Delta^+}(z^\alpha X_\alpha + \bar{z}^\alpha X_{-\alpha})} \left(\mathbf{a}\cdot\mathbf{H}\right) e^{-i\sum_{\alpha\in\Delta^+}(z^\alpha X_\alpha + \bar{z}^\alpha X_{-\alpha})},$$

where $\mathbf{a}\cdot\mathbf{H}$ is an arbitrary element of the Cartan subalgebra. Suppose that the systems is initially in an eigenstate $|m\rangle$ of $\mathbf{H}$, i.e. $\mathbf{H}|m\rangle = \mathbf{m}|m\rangle$ holds. Now, I will be concerned with a class of the systems with a cranked Hamiltonian. It is such a time dependent Hamiltonian that arise from the initial one by the adjoint action of a one-parameter group, i.e.

$$H(t) = e^{-i\omega t\mathbf{n}\cdot\mathbf{H}} H_0 e^{i\omega t\mathbf{n}\cdot\mathbf{H}},$$

where ω is called the cranking rate and $\mathbf{n}$ the cranking direction.

For such systems, the time dependent Schrödinger equation is exactly solvable. It is due the fact that through a unitary transformation $|\psi(t)\rangle = e^{-i\omega t\mathbf{n}\cdot\mathbf{H}}|\eta(t)\rangle$, we turn to the intrinsic frame. the evolution of $|\eta(t)\rangle$ is now governed by the time independent Hamiltonian of the form

$$H(\omega) = H_0 - \omega\mathbf{n}\cdot\mathbf{H} = \epsilon\left(\sum_\alpha \beta_\alpha E_\alpha + \sum_i \left(\beta_i - \frac{\omega}{\epsilon}n_i\right) H_i\right).$$

Then the state $|\psi(t)\rangle$ of the system evolves according to

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle = e^{-i\omega t\mathbf{n}\cdot\mathbf{H}} e^{-iH(\omega)t}|\psi(0)\rangle.$$

Now, using the structure of semisimple compact algebras one compute the final expression for the geometric phase [27] to be

$$\varphi_g = -2\pi\mathbf{n}\cdot\mathbf{m}\left(1 - \frac{\langle\eta_m|\mathbf{n}\cdot\mathbf{H}|\eta_m\rangle}{\mathbf{n}\cdot\mathbf{m}}\right).$$

This reveals that the phase is related to the expectation value of operators from Cartan algebra along the cranking direction $\mathbf{n}$. Further, it depends on the geometry of parameter spaces $\mathbf{n}, \mathbf{m}$, on the ray η_m generated by the Hamiltonian and on the cranking rate ω.

Let me come back to the example of a spin $\frac{1}{2}$-particle in a rotating magnetic field. It is obvious that this example belongs to the class with a cranked Hamiltonian. In this example, it is possible to proceed in a bit different direction (as in [28]). Once we have constructed $H(\omega)$, we can define the map $F : S^2 \to S^2$ from the previous chapter by setting $H(\omega) \propto H(F(x))$. Thus it is given by equations

$$S^2 \ni (\theta, \varphi) \mapsto (\tilde{\theta}, \tilde{\varphi}) \in S^2$$

$$\cos\tilde{\theta} = \frac{b}{\tilde{\omega}}(\cos\theta - \frac{\omega}{b}), \qquad \sin\tilde{\theta} = \frac{b}{\tilde{\omega}}\sin\theta, \qquad \tilde{\varphi} = \varphi.$$

But, one has to investigate if such defined map F is smooth. It need not be even single valued. For this case, it is a diffeomorphims for every ω unless the Larmor frequency. Thus one obtains immediately the result

$$\tilde{A} = -k(1 - \cos\tilde{\theta})d\varphi$$

and the Berry phase is obtained from holonomy.

Chapter 6

Conclusion

In the first part of my work, I have introduced an example from the classical physics. Namely the example of the rotating globe. Although it does not refer to the quantum mechanics, this example is quite illustrative. The phenomenon, which here appears, is very similar to the one in quantum mechanics. Namely, the additional angle of rotation, referred to as the geometric phase, arises due to the parallel transport of the instantaneous basis around a loop on the sphere which represents the globe. In fact, it is a consequence of the curvature of sphere.

In the following, I have introduced the Berry's, adiabatic, concept of the geometric phase and then the universal, Aharon-Anandan's, approach. The geometric phases, defined in different way, are shown to be the same in the adiabatic limit. The construction of Aharon and Anandan has deep theoretical implications, but do not provide any calculational useful formulas, because one has to solve the time independent Schrödinger equation. In the case of Berry phase, there are such formulas accessible, but, on the other hand, the adiabatic conditions are never exactly fulfilled.

Then, I have mentioned several experiments, which manifest the geometric phase in rather different ways. In the case of coiled optical fibre, the local curvature of the parameter space plays the crucial role and in the case of Aharon-Bohm effect, the nontriviality of the fundamental group of parameter space is responsible for the additional phase factor. The example of three-level system is totally different in that the quantum state do not evolve according to the Schrödinger dynamics, but is made to evolve by adjusting some parameters of interferometer. To conclude this part of my work, I have introduced some applications of the geometric phase.

I have devoted the most effort to the fourth chapter, where I have focused on the geometrical interpretation of the geometrical interpretation. There is a nice similarity to the motivational example. Now, in he quantum mechanics, the geometric phase arise due to the parallel transport of the instantaneous basis of Hilbert space. It is precisely the difference between the initial and the transported basis, which is known as the holonomy. In the geometric interpretation, the correspondence between the phase found by Berry and the one found by Aharon and Anandan becomes evident. For a given parameter space, the possible bundles over it were classified. On the other hand, when starting with a (suitable) Hamiltonian, the possible spaces of parameters were found. For the adiabatic case, I have further simplified the formulas for the geometric phase using the structure of Lie algebras. Finally, I have showed that the Berry's derivation can be used also in some nonadiabatic cases. As a conclusion, I have introduced several simple examples, which illustrate the developed formalism.

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